Classical Mechanics

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Exercise sheet 4 **Due: November, 9 at 12:00**

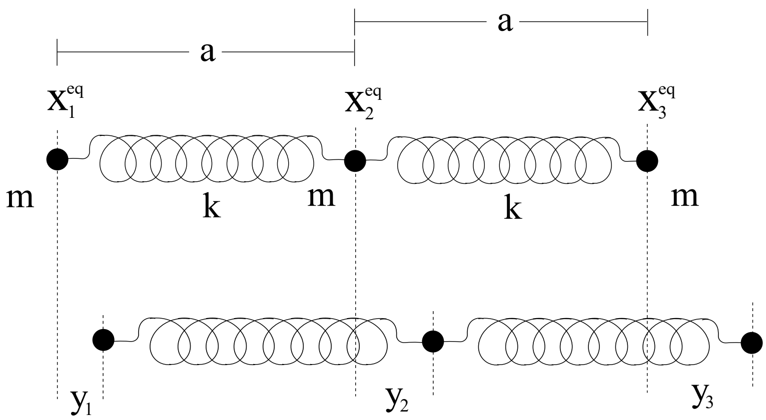
1 The Laplace-Runge-Lenz vector

You have heard about conserved quantities such as energy, momentum, and angular momentum, but sometimes a system can also possess more 'exotic' types of conserved quantities. An example turns out to be the Kepler problem, i.e., the motion of a particle in a potential of the form $V(\vec{r}) = -\alpha/r$, for $r = \|\vec{r}\|$, and for some constant α. Like for any central symmetric potential, the Kepler problem conserves the total energy and the angular momentum with respect to the center of the potential. However, it turns out that it also possesses an additional 'accidental' conserved quantity, namely the Laplace-Runge-Lenz vector, which is defined as

$$\vec{A} = \vec{p} \times \vec{L} - m\alpha \frac{\vec{r}}{r}.$$

Show that $\vec{A}$ is conserved, i.e., show that $\frac{d\vec{A}}{dt} = 0$.
Hint: Keep in mind that $\vec{L}$ is conserved. Make use of the general relation $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$. Show that $\vec{r} \cdot \frac{d\vec{r}}{dt} = r\frac{dr}{dt}$. **(4 points)**

2 Normal modes



The motion of particles that interact harmonically can be decomposed into especially simple components, the *normal modes*. For each such normal mode, the particles perform a collective periodic motion with respect to one single frequency.

As an example we consider a model of a linear molecule consisting of three atoms, each of mass m that interact according to the potential

$$V(x_1, x_2, x_3) = \frac{k}{2}(x_2 - x_1 - a)^2 + \frac{k}{2}(x_3 - x_2 - a)^2,$$

where $k > 0$ and $a > 0$. For the sake of simplicity we here only consider the motion of the atoms along the axis of the molecule.

This molecule is in mechanical equilibrium when the positions of the atoms are such that $x_1 = x_1^{\mathrm{eq}}$, $x_2 = x_2^{\mathrm{eq}}$, and $x_3 = x_3^{\mathrm{eq}}$, where $x_2^{\mathrm{eq}} - x_1^{\mathrm{eq}} = a$ and $x_3^{\mathrm{eq}} - x_2^{\mathrm{eq}} = a$. It is very convenient to introduce new coordinates $y_1 = x_1 - x_1^{\mathrm{eq}}$, $y_2 = x_2 - x_2^{\mathrm{eq}}$, and $y_3 = x_3 - x_3^{\mathrm{eq}}$ that measure the *deviation* from equilibrium. With respect to these new coordinates, we can write the potential as

$$V(y_1, y_2, y_3) = \frac{k}{2}(y_2 - y_1)^2 + \frac{k}{2}(y_3 - y_2)^2 = \frac{1}{2}\vec{y}^t \boldsymbol{V}\vec{y}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \tag{1}$$

where $\boldsymbol{V}$ is a 3×3 matrix, and t denotes the transpose, i.e., $\vec{y}^t = [y_1, y_2, y_3]$. One can also realize that $\ddot{y}_1 = \ddot{x}_1$, $\ddot{y}_2 = \ddot{x}_2$, and $\ddot{y}_2 = \ddot{x}_3$.

a) Before we start to investigate the molecule, let us first look at some more general questions concerning how to find normal modes. Suppose that

$$U(y_1, y_2, \ldots, y_N) = \frac{1}{2}\vec{y}^t \boldsymbol{U}\vec{y}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \tag{2}$$

for some matrix $\boldsymbol{U}$. *Show that one always can find a symmetric matrix $\boldsymbol{U}'$ such that $U(y_1, y_2, \ldots, y_N) = \frac{1}{2}\vec{y}^t \boldsymbol{U}'\vec{y}$.*

Hint: Recall that a matrix $\boldsymbol{U}'$ is symmetric if $\boldsymbol{U}'_{kl} = \boldsymbol{U}'_{lk}$. **(1 points)**

b) Suppose that the function U defined in (2) is the potential of a collection of particles of mass m. *Show that the equation of motion can be written*

$$m\frac{d^2}{dt^2}\vec{y} = -\boldsymbol{U}\vec{y}. \tag{3}$$

(1 points)

c) In order to find the normal modes one can make the ansatz $\vec{y}(t) = e^{i\omega t}\vec{q}$, where $\vec{q}$ is a time-independent vector, and ω a real number. *Show that this leads to an eigenvalue problem. For a given eigenvalue, what are the allowed values of the frequency ω? Write down the corresponding solutions to (3). How can you combine these solutions in order to guarantee that the resulting function is real-valued (and thus describe an actual physical motion)?*

Hint: You can use the fact that if $\boldsymbol{U}$ comes from a harmonic potential, then the eigenvalues of $\boldsymbol{U}$ always have to be real and non-negative. (This follows from $\boldsymbol{U}$ being positive semi-definite.)

(3 points)

d) Unfortunately the type of ansatz in c) does not give all solutions in the case that $\vec{q}$ is an eigenvector corresponding to eigenvalue 0. In this case we can instead try an ansatz on the form $\vec{y} = f(t)\vec{q}$ in (3). *Show that this leads to a differential equation that f has to satisfy. What is the general solution to this differential equation? Write down the corresponding solutions to (3).* **(2 points)**

e) After these general considerations, let us now turn to the linear molecule defined by the potential V. *Determine the symmetric matrix $\boldsymbol{V}$ in (1).* **(2 points)**

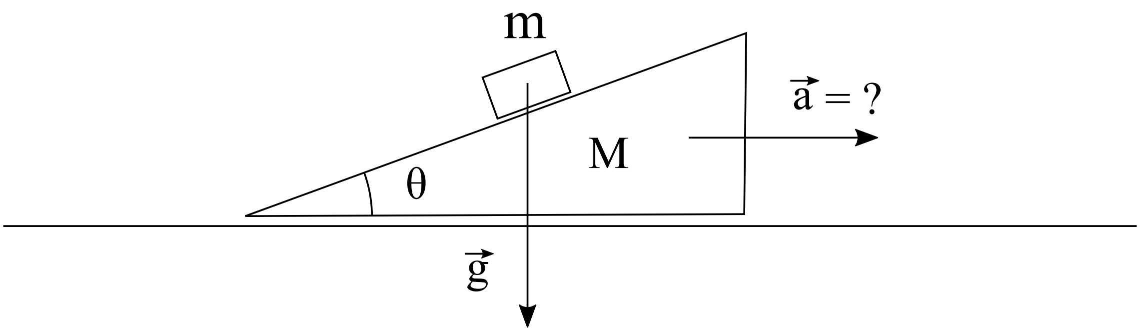
f) *Find the eigenvalues and corresponding eigenvectors of the matrix $\boldsymbol{V}$ that you found in e).* **(3 points)**

g) *Sketch the motion of the normal modes of the linear molecule. What kind of motion does the zero eigenvalue correspond to? What conservation law is it related to?* **(2 points)**

h) *Write down the general solution of the equations of motion of the linear molecule in terms of its normal modes. Make sure that the functions that you write down are real-valued.* **(2 points)**

3 A pointless challenge

This somewhat challenging exercise gives absolutely no points. However, if you hand in a reasonable solution you will get a gold star on your exercise sheet![1]

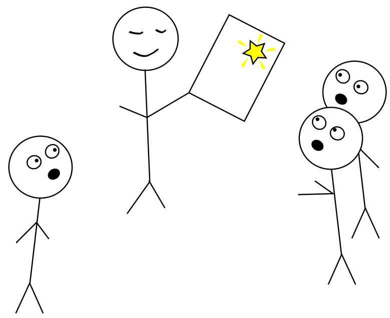


A wedge with mass M can slide without friction along a floor, and on the wedge there is a block of mass m that also can slide without friction. Imagine that we initially hold both the block and the wedge at rest, and then suddenly release them. The block is affected by gravity, and by a normal force from the wedge. (The wedge is kept on the surface by a normal force from the floor.)

Use forces and Newton's second law to determine the acceleration of the wedge.

(0 points, but a gold star!)

Hint: Think of the forces acting on the wedge, and use Newton's second law. Keep in mind that the wedge applies a normal force on the block. What does Newton's third law say concerning the force from the block on the wedge? Next, think of the forces acting on the block and again use Newton's second law to get a relation. What kind of acceleration of the block is compatible with the block sliding along the surface of the wedge? You may end up with a system of equations.



[1]Who cares about stupid points when one can get a gold star?!? Just imagine how you can impress with it in the bars along Zülpicher Straße.