Classical Mechanics

David Gross, Johan Åberg, Markus Heinrich

Exercise sheet 9 **Due: December, 14 at 12:00**

1 From Lagrange to Hamilton

For the following Lagrangians, derive the Hamilton functions, and the Hamilton equations.

a) From sheet 6, problem 1: Bead on a rotating hoop

$$L(\theta,\dot{\theta}) = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}R^2\Omega^2\sin^2\theta + mgR\cos\theta$$

(2 points)

b) From sheet 7, problem 3: Cautionary tale

$$L(r,\theta,\dot{r},\dot{\theta}) = \frac{m}{2}(1+\frac{\alpha^2}{r^6})\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 + \frac{mg\alpha}{2r^2}$$

(3 points)

2 Rationalizing the translation

As you may have noticed in the previous exercise, it can be a bit tedious to go through the procedure to translate the Lagrangian into a Hamilton function again and again. However, in practice, certain structures often reoccur, and for these we can do the translation once and for all, so to speak. In many cases (but certainly not always) the Lagrangians have the form

$$L(\vec{q},\dot{\vec{q}},t) = T(\vec{q},\dot{\vec{q}}) - V(\vec{q},t), \qquad T(\vec{q},\dot{\vec{q}}) = \frac{1}{2}\sum_k f_k(\vec{q})\dot{q}_k^2,$$

where $f_k(\vec{q})$ are some functions of the generalized coordinates $\vec{q} = (q_1,\dots,q_N)$, and where $V(\vec{q},t)$ is some possibly time-dependent potential.

Show that the Hamilton function can be written

$$H(\vec{q},\vec{p},t) = \frac{1}{2}\sum_k \frac{p_k^2}{f_k(\vec{q})} + V(\vec{q},t),$$

where $\vec{p} = (p_1,\dots,p_N)$ are the conjugate momenta with respect to $\vec{q} = (q_1,\dots,q_N)$. **(3 points)**

3 Particle in a time-dependent potential

A particle of mass m is restricted to move along a straight line. We take the coordinate q as the position of the particle along the line. The particle is affected by the time-dependent potential $V(q,t) = C\cos(\omega t)q$ for some constants C and ω.

a) *Write down the Lagrangian and the Hamilton function of the particle.* **(2 points)**

b) *Derive the Hamilton equations and solve them for the initial condition $q(0) = 0$ and $p(0) = p_0$.*

(2 points)

4 Particle in an electromagnetic field

For a particle with mass m, charge e, and position $\vec{r}$, which moves in an electromagnetic field, the Lagrangian can be written

$$L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\dot{\vec{r}}^2 - e\phi(\vec{r},t) + e\vec{A}(\vec{r},t)\cdot\dot{\vec{r}}, \tag{1}$$

where $\phi(\vec{r},t)$ is a function (the scalar potential), and a vector-valued function $\vec{A}(\vec{r},t)$ (the vector-potential)[1].

Please do not panic if you are not familiar with electromagnetism and vector-potentials, think of (1) as simply being yet another (maybe strange looking) Lagrangian of a particle.

a) Introduce the Cartesian coordinates $\vec{r} = (r_1, r_2, r_3)$ and $\vec{A} = (A_1, A_2, A_3)$. *Show that the Euler-Lagrange equations obtained from the Lagrangian (1) can be written*[2]

$$m\ddot{r}_j + e\left(\frac{\partial\phi}{\partial r_j} + \frac{\partial A_j}{\partial t}\right) - e\sum_{k=1}^{3}\dot{r}_k\left(\frac{\partial A_k}{\partial r_j} - \frac{\partial A_j}{\partial r_k}\right) = 0, \quad j = 1,2,3. \tag{2}$$

Hint: It can be useful to first rewrite (1) in terms of the Cartesian components $L(\vec{r},\dot{\vec{r}}) = \frac{m}{2}\sum_{k=1}^{3}\dot{r}_k^2 - e\phi(\vec{r},t) + e\sum_{k=1}^{3}A_k(\vec{r},t)\dot{r}_k$.

(2 points)

b) Let $\chi(\vec{r},t)$ be a real-valued function (a scalar function), and suppose that we change ϕ and $\vec{A}$ into the new functions ϕ' and $\vec{A}'$ by

$$\phi' = \phi - \frac{\partial\chi}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla\chi.$$

Let L' be the new Lagrangian that is obtained if we substitute ϕ and $\vec{A}$ in (1) by ϕ' and $\vec{A}'$. *Show that L and L' only differ by a total time-derivative of some function of $\vec{r}$ and t.* **(2 points)**

Comment: This type of mapping of the potentials is called a gauge-transformation. Recall that the addition of a total time-derivative to the Lagrangian does not change the Euler-Lagrange equations (and thus does not change the evolution of the system).

c) *Derive the Hamilton function.* **(2 points)**

d) *Derive the Hamilton equations.* **(2 points)**

[1] The electric field can be obtained as $\vec{E}(\vec{r},t) = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}$ and the magnetic field as $\vec{B}(\vec{r},t) = \nabla\times\vec{A}$.
[2] The more standard way of writing (2) is $m\ddot{\vec{r}} = e\vec{E} + e\dot{\vec{r}}\times\vec{B}$. The right hand side is the so-called *Lorentz force*.

5 Rationalizing the translation again

This exercise gives no points, but a gold star! The derivation that we did in exercise 2 can be generalized to a more general case. Consider Lagrangians on the form

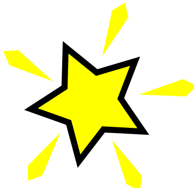
$$L(\vec{q},\dot{\vec{q}},t) = \frac{1}{2}\sum_{jk} \boldsymbol{F}(\vec{q},t)_{jk}\dot{q}_j\dot{q}_k + \sum_k g_k(\vec{q},t)\dot{q}_k - V(\vec{q},t),$$

where $\boldsymbol{F}(\vec{q},t)$ is a symmetric matrix for each $\vec{q}$, t, and where $g_k(\vec{q},t)$ are functions.

Show that

$$H(\vec{q},\vec{p},t) = \frac{1}{2}\sum_{jk}[\boldsymbol{F}(\vec{q},t)^{-1}]_{jk}\Big(p_j - g_j(\vec{q},t)\Big)\Big(p_k - g_k(\vec{q},t)\Big) + V(\vec{q}),$$

where $\boldsymbol{F}(\vec{q},t)^{-1}$ denotes the matrix inverse of $\boldsymbol{F}(\vec{q},t)$[3]. **(0 points, but a gold star!)**



[3] Strictly speaking, this only works for points $(\vec{q},t)$ where $\boldsymbol{F}(\vec{q},t)$ is invertible. We implicitly assume something similar in exercise 2, when we write things like $\frac{1}{f_k(\vec{q})}$.