Classical Mechanics

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Exercise sheet 6 **Due: November, 22 at 12:00**

1 Block and wedge again

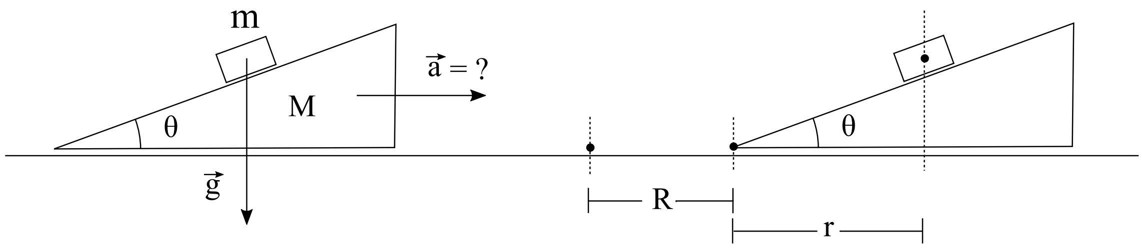

Figure 1: A wedge of mass M can slide without friction along a horizontal surface. On the wedge there is a block of mass m that slides along the surface of the wedge without friction. The block is affected by gravity. The coordinate r is the horizontal distance from the edge of the wedge to the center of mass of the block. R is the distance from the edge of the wedge to some reference point on the plane.

A wedge can slide without friction along a floor, and on the wedge there is a block that also can slide without friction. Imagine that we initially hold both the block and the wedge still, and then suddenly release them.

a) *Derive the Lagrange function for the wedge and block with respect to the coordinates r and R described in the figure.*[1] **(2 points)**

b) *Use the Lagrange function to obtain the Euler-Lagrange equations.* **(2 points)**

c) *Use the Euler-Lagrange equations to find the acceleration of the wedge.* **(2 points)**

d) When you solved b) you may have noted that $\frac{\partial L}{\partial R} = 0$.[2] By the Euler-Lagrange equation it follows that $\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} = 0$. This means that $\frac{\partial L}{\partial \dot{R}}$ is a time-independent quantity, i.e., it is conserved. *What conserved quantity does $\frac{\partial L}{\partial \dot{R}}$ correspond to? Can you explain, in terms of forces, why this quantity is conserved?* **(2 points)**

Comment: This is the first exercise where we apply the Lagrangian method. Compared to the Newtonian approach, where one has to keep track of forces (think of exercise 1.3), the Lagrangian method tends to be less messy.

[1] The beauty of the Lagrange method is that one can use whichever coordinates one wants. However, let us nevertheless settle for this choice of coordinates so that we do not drive the poor tutors to the brink of tears. (They would have to check each new derivation for each and every choice of coordinates that you could come up with.)

[2] When $\frac{\partial L}{\partial R} = 0$ the coordinate R is often referred to as being 'cyclic'.

2 Optimizing happiness

Figure 2: Students Alice and Bob honoring the local traditions.

Students Alice and Bob keenly participate in the Cologne local traditions, and want to make sure that they optimize their accumulated level of happiness on the 11th of November. In the course on classical mechanics that they are currently taking, they have just learned about the Lagrangian method. They know that one can obtain the equations of motions of physical systems by finding stationary solutions to the functional $\mathcal{S}[x] = \int_{t_i}^{t_f} L(x, \dot{x}, t)dt$ for the Lagrange function L. They also know that they can find such stationary solutions by solving the Euler-Lagrange equation corresponding to L[3]. Clever as they are, they realize that this method can be used for optimizing all sorts of things, and they set forth to celebrate 11th of November with scientific precision.

Alice and Bob are totally convinced that their level of happiness at each given time t roughly can be equated with the accumulated amount $x(t)$ of various liquid refreshments that they have imbibed up to time t (counted from $t_i = 11{:}11$). However, they also agree that a too rapid intake leads to a lot of messy and unpleasant side-effects, and that the speed of the intake, $\dot{x}$, is most comfortable around $\alpha > 0$. After several experiments they agree that the following expression gives a good estimate of their level of happiness at any given instant of time[4]

$$L_{\text{happiness}}(x, \dot{x}) = x\dot{x}(2\alpha - \dot{x}).$$

Hence, they wish to find the function $x(t)$ that maximizes their accumulated happiness from $t_i = 11{:}11$ to $t_f =$ very very late. In other words, they want to optimize the happiness functional

$$\mathcal{S}_{\text{happiness}}[x] = \int_{t_i=11:11}^{t_f=\text{very very late}} L_{\text{happiness}}(x, \dot{x})dt. \tag{1}$$

The solution $x(t)$ must satisfy some boundary conditions, namely that they do not start to drink before $t_i = 11{:}11$, and thus $x(t_i) = 0$. Moreover, they decide to in total drink the quantity Q (e.g. due to budget constraints). Hence, $x(t_f) = Q$.

a) Before we start to analyze Alice and Bob's very important problem, let us first make a general observation. Suppose that we have some function $L(x, \dot{x}, t)$ and define the new function[5]

$$H(x, \dot{x}, t) = \dot{x}\frac{\partial L}{\partial \dot{x}} - L(x, \dot{x}, t).$$

[3] Please do not panic about these functionals. You only need the Euler-Lagrange equation in order to solve this problem.

[4] One can certainly question the realism of this model, but who are we to argue with Alice and Bob's perception of their own happiness.

[5] This type of function will play an important role later in the course.

Suppose that L is such that $\frac{\partial L}{\partial t} = 0$. Show that this implies that $\frac{dH}{dt} = 0$ whenever $x(t)$ is a solution to the Euler-Lagrange equation with respect to L. In other words, show that for every solution $x(t)$, there exists some constant C such that $H(x(t), \dot{x}(t), t) = C$.

Hint: Note the difference between the partial derivative $\frac{\partial L}{\partial t}$ and the total derivative $\frac{dL}{dt}$. The function L not only depends on t, but also on x and $\dot{x}$, which in turn depend on t. When we write $\frac{\partial L}{\partial t}$ we only differentiate the *direct* dependence on t in $L(x, \dot{x}, t)$, while keeping x and $\dot{x}$ fixed. However, when we write $\frac{dL}{dt}$ we differentiate with respect to the total dependence on t in the function $L(x(t), \dot{x}(t), t)$. Use the chain rule to rewrite the total derivative $\frac{dL}{dt}$. **(3 points)**

b) The observation in a) can sometimes be used to solve the equations of motion. The Euler-Lagrange equation typically gives a differential equation that is second order in time, while the relation $H = C$ gives an equation that is first order in time, which might be easier to solve. *Use this in order to find a stationary solution to* (1).[6] In this problem you will encounter various roots. You are allowed to simply pick the sign of constants, and the roots, such that you get real positive solutions[7]. **(4 points)**

c) *Find the Euler-Lagrange equation corresponding to* (1)*, and confirm that your solution in b) actually solves that Euler-Lagrange equation.* **(3 points)**

d) When Alice and Bob derive the solution $x(t)$ to the Euler-Lagrange equation, they find that it does not depend on α. This make them a bit confused, so they plug the solution into $\mathcal{S}_{\text{happiness}}$ (which does depend on α). To their horror they find that it can become negative for some values of α, $\Delta t = t_f - t_i$, and Q[8]. *Determine $\mathcal{S}_{\text{happines}}[x]$ in terms of α, Δt, and Q. By keeping α and Δt fixed, what is the maximal value $\mathcal{S}^* = \max_{Q \geq 0} \mathcal{S}_{\text{happiness}}[x]$, and at which value Q^* is it attained?*

Let us now compare the optimal choice with the choice to constantly drink at the most comfortable speed α, i.e., $x_{\text{comfort}}(t) = \alpha(t - t_i)$. *How does Q^* compare with the total intake for x_{comfort}? Calculate $\mathcal{S}^* / \mathcal{S}_{\text{happiness}}[x_{\text{comfort}}]$.*[9] **(2 points)**

Comment: The purpose of this exercise is to illustrate that the technique (variational calculus) by which we obtain the Euler-Lagrange equations, can be applied to many other things than mechanics.

[6] Alice and Bob of course want to find the maximum. Unfortunately, it can often be rather difficult to prove that the stationary solution actually is a minimum or maximum. One often simply has to live with this.

[7] Whatever $x(t) < 0$ or $\dot{x}(t) < 0$ would mean, it would certainly not be something pleasant, so we do not want to think about it.

[8] This is maybe not that strange. By the boundary condition $x(t_f) = Q$, they have to consume the amount Q over the time Δt. Hence if Q is very large, or Δt very small, then they have to go far beyond their optimal drinking speed α, and L can become negative.

[9] Hmmm... I suddenly realize that the moral of this tale got a bit dubious. I had hoped to get something about the virtues of moderate drinking. Well, shit happens. It's carnival!