Classical Mechanics

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Handy formulas

- **Noether's theorem**. Suppose the Lagrangian is transforming under a transformation $\vec{q}_i \mapsto \vec{q}_i' = \vec{h}_i^\varepsilon(\vec{q}_1, \ldots, \vec{q}_n)$ as $L \mapsto L + \frac{d}{dt} f$. Then, the quantity

$$J = \frac{\partial f}{\partial \varepsilon}\Big|_{\varepsilon=0} - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{\vec{q}}_i} \cdot \frac{\partial \vec{h}_i^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

  is conserved.

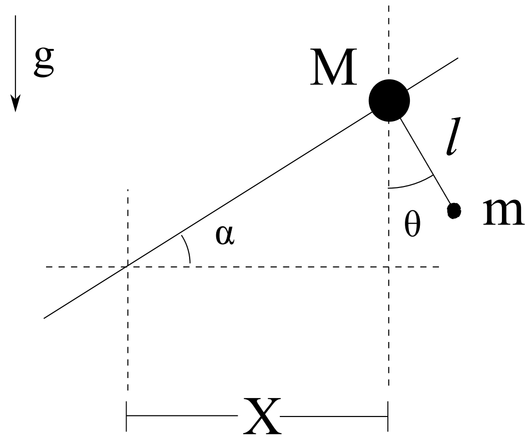
- **Poisson brackets**

$$\{f, g\} = \sum_{n=1}^{N} \left(\frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n} - \frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} \right).$$

- Equation of motion of a point particle of mass m in a **rotating frame** $\vec{r}'$. Here, $\vec{R}$ is the position vector which points to the origin of the rotating frame, $\vec{\omega}$ is the angular velocity vector of the rotation and $\vec{F}$ an external force.

$$m\ddot{\vec{r}}' = \vec{F} + m\ddot{\vec{R}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') - 2m\vec{\omega} \times \dot{\vec{r}}'.$$

1 Lagrangian, and the Euler-Lagrange equations



A mass M slides without friction along a straight rail that is tilted at a fixed angle α with respect to the horizontal plane. From the mass M hangs a pendulum in the form of a massless rod of length l to which a mass m is attached. The pendulum only swings in the plane of the vertical axis and the rail. Both the mass M and m are affected by the constant gravity g in the vertical direction.

a) Let X be the position of M along the horizontal plane, and let θ be the angle between the vertical axis and the pendulum *Derive the Lagrangian with respect to the coordinates X and θ.*

b) *Obtain the Euler-Lagrange equations for the motion of the particle.*

2 From Lagrange to Hamilton, and the Hamilton equations

Consider the following Lagrangian that describes the motion of a particle with mass m

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{m}{2}(\dot{x} - \omega y)^2 + \frac{m}{2}(\dot{y} + \omega x)^2 + \frac{m}{2}\dot{z}^2.$$

a) *Determine the conjugate momenta corresponding to x, y, and z.*

b) *Derive the corresponding Hamilton function.*

c) *Derive the Hamilton equations.*

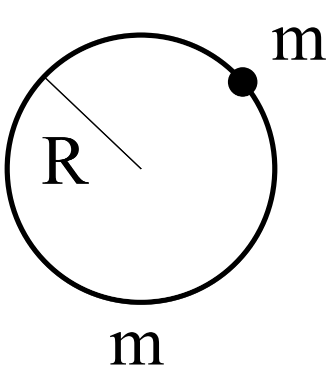
3 Canonical transformations

Consider the mapping from (q, p) to (Q, P) defined by

$$Q = \ln\Big(1 + q^\alpha \cos(\beta p)\Big), \quad P = 2\Big(1 + q^\alpha \cos(\beta p)\Big) q^\alpha \sin(\beta p).$$

For which values of the constants α and β is this a canonical transformation?

4 Inertia tensor



Consider an infinitely thin ring of mass m and radius R. On the ring there is an additional point mass m attached (hence the total mass is $2m$). *Determine the inertia tensor with respect to the principal axis system with origin at the center of mass.*

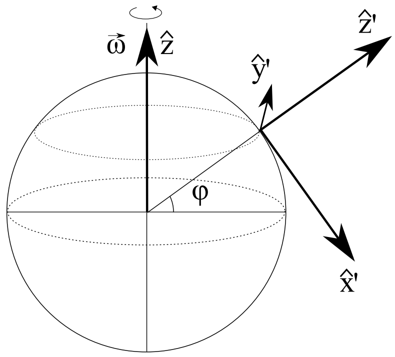
Hint: The moment of inertia with respect to an axis that *lies in the plane* of the ring, and goes through its center, is $I_{\text{plane}} = \frac{1}{2}mR^2$.

5 Phase space flow

Suppose that a particle of mass m moves on a straight line and is affected by the potential $V(x) = \alpha x^2 - \beta x^4$, where $\alpha > 0$ and $\beta > 0$.

Sketch the flow in the phase space of the particle. You do not have to do a very detailed or exact picture, a rough sketch is enough. *Indicate in the figure what the equilibrium points are, and which that are stable and which are unstable.*

6 Newton, apples and a spinning earth



Although Newton hates apples, he is once again sitting next to a apple tree. He knows that the earth is actually rotating at this very moment and wonders how that will affect the trajectory of apples falling from that tree. Consider a rotating coordinate system $\hat{x}', \hat{y}', \hat{z}'$ with origin somewhere on the surface of the earth with latitude φ. We will assume that the axis are oriented such that $\hat{z}'$ is pointing radially outwards (i.e. vertically upwards with respect to the surface), $\hat{x}'$ is pointing south, and $\hat{y}'$ is pointing east.

a) *Express the angular velocity vector of earth, $\vec{\omega} = \omega\hat{z}$, in the rotating frame.*

b) *Write down the equations of motion for a apple falling from height h in the rotating frame and solve them.* Assume that the rotation of the earth is actually slow such that you can ignore terms of order ω^2 or higher.

c) *How large is the horizontal deviation (i.e. the one in $\hat{y}'$ direction) when the apple hits the ground compared to one that would fall in a straight line? For which angle φ the deviation the largest?*

7 Noether's theorem

Consider a system of N particles as follows

$$L = \sum_{i=1}^{N} \frac{m_i}{2} \dot{\vec{r}}_i^2 - \frac{1}{2} \sum_{i,j=1}^{N} V(\vec{r}_i - \vec{r}_j).$$

Show that the system is invariant under Galilei transformations

$$\vec{r}_i \mapsto \vec{r}_i' = \vec{r}_i + \varepsilon \vec{v} t,$$

for arbitrary velocities $\vec{v}$.

Derive the conserved quantity from Noether's theorem. What is the physical meaning of that conservation?