14756: Quantum Error Correction **WS 2018**

Solution 3

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Excercise 1

Solution to 1.1 Matrices $\{\mathbb{1}, X, Y, Z\}$ form a basis of the vectorspace of 2×2 matrices over $\mathbb{C}$ with complex coefficients. In other word, any 2×2 matrix A can be written as $A = \sum_{\mu=0}^{4} \alpha_\mu \sigma_\mu$ where $\sigma_0 = \mathbb{1}$, σ_i are Pauli matrices for $i \in \{1, 2, 3\}$ and $\alpha_\mu \in \mathbb{C}$. Using the orthogonality of Pauli matrices, we can also write $\alpha_\mu = \mathrm{Tr}[\sigma_\mu A]/2$.

Using this result, we also can decompose error E_k as $E_k = \sum_\mu e_\mu^k \sigma_\mu$. The constraint of this channel is $\sum_k E_k^\dagger E_k = \mathbb{1}$ which is the trace preserving condition. Calculating this, we obtain

$$\sum_k E_k^\dagger E_k = (\overline{e_0^k}\mathbb{1} + \sum_i \overline{e_i^k}\sigma_i)(e_0^k \mathbb{1} + \sum_j e_j^k \sigma_j) \tag{1}$$

$$= \overline{e_0^k} e_0^k \mathbb{1} + \sum_{i,j} \overline{e_i^k} e_j^k \sigma_i \sigma_j + \sum_i \overline{e_i^k} e_0^k \sigma_i + \sum_j \overline{e_0^k} e_j^k \sigma_j \tag{2}$$

$$= \sum_\mu \overline{e_\mu^k} e_\mu^k \mathbb{1} + \Big[\sum_{i,j} i\overline{e_i^k} e_j^k \epsilon_{ijl} + \overline{e_l^k} e_0^k + \overline{e_0^k} e_l^k\Big]\sigma_l, \tag{3}$$

where we used $\sigma_i \sigma_j = \delta_{i,j} + i\epsilon_{ijk}\sigma_k$ and $\overline{a}$ is a complex conjugate of a. Thus we require $\overline{e_\mu^k} e_\mu^k = 1$ and $\sum_{i,j} i\overline{e_i^k} e_j^k \epsilon_{ijl} + \overline{e_l^k} e_0^k + \overline{e_0^k} e_l^k = 0$.

Solution to 1.2 Recall that logical qubit of the Shor code is given as $|0_L\rangle = |+++\rangle$ and $|1_L\rangle = |---\rangle$ where $|\pm\rangle = (|000\rangle \pm |111\rangle)/\sqrt{2}$. Knil-Laflamme theorem states that $\{E_k\}$ are correctable if and only if $PE_i^\dagger E_j P = \alpha_{i,j} P$ for some $\alpha_{i,j}$ that is a Hermitian matrix (i.e., $\alpha_{i,j} = \alpha_{j,i}^*$). Here, P is a projector on the logical qubit subspace. In our case, $P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$. Putting this P, we obtain

$$\begin{aligned} PE_i^\dagger E_j P = &|0_L\rangle\langle 0_L| \langle 0_L|E_i^\dagger E_j|0_L\rangle + |0_L\rangle\langle 1_L| \langle 0_L|E_i^\dagger E_j|1_L\rangle \\ &+ |1_L\rangle\langle 0_L| \langle 1_L|E_i^\dagger E_j|0_L\rangle + |1_L\rangle\langle 1_L| \langle 1_L|E_i^\dagger E_j|1_L\rangle. \end{aligned} \tag{4}$$

For single qubit errors $\{E_k\}$, it is easy to show that $\langle 0_L|E_i^\dagger E_j|1_L\rangle = \langle 1_L|E_i^\dagger E_j|0_L\rangle = 0$. Likewise, we also obtain $\langle 0_L|X_a|0_L\rangle = \langle 0_L|Y_a|0_L\rangle = \langle 0_L|Z_a|0_L\rangle = 0$ and $\langle 1_L|X_a|1_L\rangle = \langle 1_L|Y_a|1_L\rangle = \langle 1_L|Z_a|1_L\rangle = 0$ where a is the location of the error. As $E_i^\dagger E_j = (\overline{e_0^i} e_0^j + \overline{e_1^i} e_1^j + \overline{e_2^i} e_2^j + \overline{e_3^i} e_3^j)\mathbb{1} + (\cdots)X + (\cdots)Y + (\cdots)Z$, $\langle 0_L|E_i^\dagger E_j|0_L\rangle = \langle 1_L|E_i^\dagger E_j|1_L\rangle = \overline{e_0^i} e_0^j + \overline{e_1^i} e_1^j + \overline{e_2^i} e_2^j + \overline{e_3^i} e_3^j = \sum_{\mu=0}^{3} \overline{e_\mu^i} e_\mu^j$. To sum up, we obtain $\alpha_{i,j} = \sum_{\mu=0}^{3} \overline{e_\mu^i} e_\mu^j$ so $\alpha_{j,i} = \sum_{\mu=0}^{3} \overline{e_\mu^j} e_\mu^i = \alpha_{i,j}^*$.

Exercise 2

Solution to 2.1 Depolarizing channel is given as $\mathcal{E}(\rho) = (1-p)\rho + p/3(X\rho X + Y\rho Y + Z\rho Z)$. Thus $\mathcal{E}^{\otimes 9}(\rho) = \sum_k (1-p)^{9-k}(p/3)^k \sum_{a=1}^{3^k\binom{9}{k}} E_a^k \rho (E_a^k)^\dagger$ where $\{E_a^k\}$ are all possible configuration of errors with k

Pauli operators. For example, $E_1^0 = \mathbb{1}^{\otimes 9}$ and E_a^1 for $a \in [1, \cdots, 27]$ are all possible single qubit errors. As there are three different errors on each qubit, we have 27 configurations. In addition, we already know that the Shor$_9$ code can correct any arbitrary single qubit error. So these 27 configurations are all correctable. Then let us consider $k = 2$ case. In this case, there are some correctable and uncorrectable errors. For example, let us consifer there are two X errors in the same cluster of qubits. Then the error correcting procedure for $X_1X_2\left|\pm\right\rangle$ makes a logical error. To be precise, we measure two syndromes Z_1Z_2 and Z_2Z_3. The first gives 1 and the second gives -1. Then the majority vote suggests correcting it by applying X_3. Then the resulting state is $X_1X_2X_3\left|\pm\right\rangle = \pm\left|\pm\right\rangle$ that means logical phase flip error in the encoded logical qubit. On the other hand, when there are two X errors in two different clusters then it is correctable. Likewise, we also see that two Z errors in the same cluster does not change the state but in different clusters makes a logical error. From this working principle, we can find the correctable error configurations as follows:

XX	XY	XZ	YY	YZ	ZZ
27	54	72	0	18	9

Here, Y error is just considered as XZ. Thus among total $\binom{9}{2}3^2 = 324$ error configurations, 180 error configurations are correctable. Likewise, we can extent this to larger k. As it is complicated to calculated all correctable configurations by hand, we instead calculate it by making a simple program. Using that, we obtain the number of correctable configurations as $\{1, 27, 180, 804, 2502, 6858, 14580, 20916, 15633, 4035\}$ from $k = 0$ to 9. It is also interesting as there are correctable error configurations even for $k = 9$. This is because two Z errors in the same cluster and X errors on two whole clusters do not change the state. Example correctable errors in this case are $Z_1Z_2X_3Z_4Z_5X_6Z_7Z_8X_9$ and $X_1X_2X_3X_5X_6X_7Z_1Z_2Z_3$.