Exercise Sheet 9

Kastoryano: Quantum Error Correction

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1 Exercise 1: Universal rotations

Let H be the Haddamard gate, and let T be the $\pi/8$ gate;

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad\qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \tag{1}$$

Show that

$$THTH \propto \exp(-i\frac{\pi}{8}Z)\exp(-i\frac{\pi}{8}X), \tag{2}$$

and argue that the combined operation corresponds to a rotation of the Bloch sphere about the axis given along $\vec{n} = (\cos(\pi/8)), \sin(\pi/8), \cos(\pi/8))$ by an angle $\theta = 2\arccos(\cos^2(\pi/8))$.

2 Exercise 2: Irrationality of θ (NC, 4.42)

Suppose $\cos\theta = 3/5$. Give a proof by contradiction that θ is an irrational multiple of 2π.

- Using the fact that $e^{i\theta} = (3 + 4i)/5$, show that if θ is rational, then there must exist a positive integer m such that $(3 + 4i)^m = 5^m$.

- Show that $(3+4i)^m = 3 + 4i(\mod 5)$ for all $m > 0$, and conclude that no m such that $(3 + 4i)^m = 5^m$ can exist.

Bonus: Can you show the same for $\theta = 2\arccos(\cos^2(\pi/8))$?

3 Injected S gate

Show that the following circuit implements the phase (S) gate, where the state $|Y_L\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$.

$$|\psi_L\rangle \quad\bullet\quad\quad\bullet\quad\quad \hat{S}_L|\psi_L\rangle$$

$$|Y_L\rangle \quad\oplus\quad H \quad\oplus\quad H \quad |Y_L\rangle$$