A pedestrian's view on interacting particle systems, KPZ universality and random matrices

2010 J. Phys. A: Math. Theor. 43 403001

(http://iopscience.iop.org/1751-8121/43/40/403001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 403001 (41pp) doi:10.1088/1751-8113/43/40/403001

TOPICAL REVIEW

A pedestrian's view on interacting particle systems, KPZ universality and random matrices

Thomas Kriecherbauer[1] **and Joachim Krug**[2]

[1] Fakultät für Mathematik, Ruhr-Universität Bochum, Germany
[2] Institut für Theoretische Physik, Universität zu Köln, Germany

E-mail: thomas.kriecherbauer@ruhr-uni-bochum.de and krug@thp.uni-koeln.de

Received 29 April 2010, in final form 28 July 2010
Published 15 September 2010
Online at stacks.iop.org/JPhysA/43/403001

Abstract

These notes are based on lectures delivered by the authors at a Langeoog seminar of SFB/TR12 *Symmetries and Universality in Mesoscopic Systems* to a mixed audience of mathematicians and theoretical physicists. After a brief outline of the basic physical concepts of equilibrium and nonequilibrium states, the one-dimensional simple exclusion process is introduced as a paradigmatic nonequilibrium interacting particle system. The stationary measure on the ring is derived and the idea of the hydrodynamic limit is sketched. We then introduce the phenomenological Kardar–Parisi–Zhang (KPZ) equation and explain the associated universality conjecture for surface fluctuations in growth models. This is followed by a detailed exposition of a seminal paper of Johansson [59] that relates the current fluctuations of the totally asymmetric simple exclusion process (TASEP) to the Tracy–Widom distribution of random matrix theory. The implications of this result are discussed within the framework of the KPZ conjecture.

PACS numbers: 05.40.−a, 02.50.Cw, 02.30.Ik, 05.70.Ln

(Some figures in this article are in colour only in the electronic version)

1. Introduction and outline

In statistical mechanics the study of systems that are far from equilibrium continues to attract considerable attention both in the physics and in the mathematics literature. As it turns out, exclusion processes first introduced by Spitzer [122] and their generalizations provide an excellent set of models that display rich and interesting nonequilibrium phenomena. Moreover, these processes are intimately related to a number of different models from statistical mechanics, combinatorics, probability and random matrix theory and a fruitful interplay between these fields, triggered by the seminal work of Baik, Deift and Johansson [6], continues to produce spectacular results that yield very precise information on the behavior of such systems.

It is beyond the scope of this review to explain all or even a fair amount of these results in detail. Instead, we remain faithful to the original goal of the set of lectures on which this manuscript is based, namely to explain the fascinating developments in this field to an audience of scientists working in many different areas of mathematics and theoretical physics. We do not assume any significant acquaintance with the concepts of statistical mechanics or probability theory. In order to keep our presentation as elementary as possible we shall always focus on the simplest cases. Readers who wish to obtain more information on a specific topic will be referred to the literature. Here we will make use of the numerous reviews that have appeared recently in this area.

In the remaining part of the introduction we provide a short outline of the topics that will be treated in this paper.

In section 2 we begin our discussion by explaining in general terms what physicists mean by the distinction between equilibrium and nonequilibrium systems, and by describing different types of nonequilibrium behavior. We then introduce in section 3 a class of stochastic models, known as *simple exclusion processes*. They describe the stochastic motion of interacting particles on a lattice where the interaction is given by the exclusion property, i.e. two particles may not occupy the same site simultaneously. Interacting particle systems provide useful models for various nonequilibrium phenomena. Technically, they are (discrete or continuous time) Markov chains, and we will argue below in section 3.2 that a simple yet precise criterion for the equilibrium versus nonequilibrium character of a given system can be formulated within the general theory of Markov chains [144].

We then proceed to explain in more detail how one-dimensional simple exclusion processes can be analyzed, specializing to the case where particles move only to neighboring lattice sites. If the probabilities to move right or left differ from each other one obtains the *asymmetric simple exclusion process* (ASEP). Mostly this review will be concerned with ASEP and its subcase TASEP (*totally asymmetric simple exclusion process*) where all motion is unidirectional. The criterion of section 3.2 identifies ASEP (and consequently TASEP) as nonequilibrium systems. In fact, ASEP has become a paradigmatic model for driven transport of a single conserved quantity and most of our discussion is focused on this class of models. For readers who are not familiar with ASEP and TASEP it might be useful at this point to have a look at the precise definition in section 3.1.

We begin our exposition of the analysis of ASEP with a discussion of stationary measures in section 3.3 where it is explained why the uniform distribution always provides a stationary measure in the simplest case of periodic boundary conditions.

For a macroscopic description of the dynamics the notion of the hydrodynamic limit is introduced. The time evolution of the macroscopic particle density is then described by a hyperbolic PDE that can be solved by the method of characteristics (section 3.4). After this has been established it is most natural to ask how much the process fluctuates around this macroscopic description. At this point it is useful to realize that ASEP is equivalent to a specific random model for surface growth which is known as 'corner growth' or 'single-step' model (section 3.5). Kardar, Parisi and Zhang (KPZ) conjectured in their seminal paper [63] that the fluctuation properties of a large class of (growth) models are universal. Assuming that KPZ universality can be applied to this particular growth model, one may obtain predictions for the scaling exponents of the fluctuations of the height of the surface and of the scaling exponent of the (spatial) correlation length. The KPZ conjecture and its implications for ASEP will be explained in section 4.

The KPZ conjecture is based on the KPZ equation which essentially adds stochastic driving to the hyperbolic PDE that describes the hydrodynamic limit. Unfortunately, the KPZ equation is difficult to analyze and it was only very recently that fluctuation results for

solutions of the KPZ equation became available (cf section 8.4). Before that an alternative approach to studying fluctuation properties was taken that turned out to be very fruitful. Rather than dealing with the KPZ equation itself one analyzes various specific models that are believed to belong to the KPZ universality class. A major breakthrough was achieved through a spectacular discovery on Ulam's problem for random permutations.

In the early 1960s Ulam raised the question to determine the asymptotics of the length of the longest increasing subsequence of a random permutation of the numbers $1, \ldots, N$, where it is assumed that all $N!$ permutations are equally likely. It took about 15 years to prove that the expected value of the length of the longest increasing subsequence behaves like $2\sqrt{N}$ as N becomes large and there was strong numerical evidence that the fluctuations around the mean are of order $N^{1/6}$. In a remarkable paper Baik, Deift and Johansson [6] did not only prove that $N^{1/6}$ was indeed the correct scaling but they also identified the limiting distribution for the appropriately rescaled fluctuations. It came as a surprise that the limiting distribution coincides with the Tracy–Widom distribution that describes the fluctuations of the largest eigenvalue of matrices from the Gaussian unitary ensemble (GUE) as the matrix size tends to infinity.

The article [6] became the starting signal for an explosion of research activities that continue until today. It became immediately clear that there are a variety of related combinatorial models (e.g. growth models, last passage percolation models, tilings, directed polymers in a random environment, tandem queues) that can be analyzed at the same level of detail and where random matrix distributions appear in the asymptotic description. For further information we refer the reader to the surveys [1, 60, 68, 88, 125, 129] and the monograph [11]. Two results that are of particular relevance for our discussion of KPZ universality were obtained independently by Prähofer and Spohn [99] and Johansson [59].

Prähofer and Spohn used the results in [6], and further developments in [10], to describe the height fluctuations of the polynuclear growth model (which is somewhat different from the corner growth/single-step model mentioned above, cf section 8). They obtained the scaling law predicted by KPZ universality. Moreover, they were able to identify the limiting distributions of the fluctuations. Again these are given by the Tracy–Widom distributions of random matrix theory. A distinction needs to be made depending on the curvature of the surface. Flat surfaces lead to statistics from the Gaussian orthogonal ensemble (GOE), whereas the height fluctuations of curved surfaces are described by GUE statistics.

On the other hand, the results that Johansson presented in [59] immediately apply to the totally asymmetric simple exclusion process with step initial condition. One obtains the scaling exponent and the limiting distribution for the fluctuations of the particle flux that are again given by the GUE Tracy–Widom distribution. We devote sections 5–7 to explain [59] in great detail. This part of our presentation can be viewed as an expanded and more self-contained version of sections 3 and 4 of the review [110] by Sasamoto.

According to the philosophy of our paper we explain the results of Johansson in the simplest case. More precisely, we consider the particle flux at the origin for a discrete time version of TASEP (dTASEP, cf section 3.1 (iii)) with step initial data. In section 5 we formulate Johansson's result in theorem 5.4 and discuss its relation to KPZ universality.

The proof of theorem 5.4 naturally falls into two parts. The first part is of combinatorial nature. Via a representation by waiting times (section 6.1) the problem is mapped to finding the longest subsequence in a list of alphabetically ordered two-letter random words that is weakly increasing in the second letter (section 6.2). By the Robinson–Schensted–Knuth algorithm (section 6.3) one may represent the random words by pairs of semistandard Young tableaux of the same shape (see definition 6.2). The advantage of this representation is twofold: on the one hand the length of the longest weakly increasing subsequence is simply

given by the length of the first row of the corresponding Young tableau. On the other hand there exist explicit formulae for counting the number of semistandard Young tableaux of a given shape that can be derived using Schur polynomials (section 6.4). The result of all this reasoning is formula (60), where Δ denotes the Vandermonde determinant (also see (44) and definitions 5.1 and 5.2).

The second part of the proof of theorem 5.4 is the asymptotic analysis of (60). The key observation is that the right-hand side of (60) has exactly the same structure as the formula for the distribution of the largest eigenvalue of GUE matrices. In particular the method of orthogonal polynomials (section 7.1) can be applied to complete the proof of theorem 5.4 in section 7.2. The somewhat miraculous appearance of the Tracy–Widom distribution for the fluctuation of the particle flux of dTASEP is now explained on a technical level by the fact that Hermite polynomials (used for GUE) and Meixner polynomials (used for dTASEP) look the same near their respective largest zeros after appropriate rescaling. The similarity of Hermite and Meixner polynomials is no coincidence. We briefly discuss the universal behavior of orthogonal polynomials in section 7.3.

As was mentioned above, the work of Johansson [59] and of Prähofer and Spohn [99] mark the beginning of a broad stream of research activities that continue to produce new and exciting results at a rapid pace. In section 8 we briefly sketch and summarize those directions of recent research that are closely related to the question of KPZ universality. The first generalization beyond [59] that we describe concerns the initial conditions. The results of Johansson apply for step initial conditions where every site to the left of the origin is occupied whereas every site to the right is empty. Based on their work on the polynuclear growth model Prähofer and Spohn [101] formulated a conjecture for the fluctuations of the flux for TASEP in the case of a general initial step profile with arbitrary constant particle densities ρ_L and ρ_R to the left resp. right of the origin. In section 8.1 we explain this conjecture that has recently been fully established by Ben Arous and Corwin [4]. Most remarkably, in a series of papers [130–136] Tracy and Widom were able to extend some of these results to general ASEP. It should be pointed out that their proof is based on the Bethe ansatz and does not use any of the nice but very special combinatorial identities that were crucial in the argument of Johansson.

Furthermore we provide in section 8 pointers to the recent literature regarding spatio-temporal correlations for (T)ASEP (section 8.2), interacting particle systems beyond (T)ASEP (section 8.3), fluctuation results for the KPZ equation (section 8.4) and physical experiments where KPZ behavior can be observed (section 8.5). We conclude the paper with a few remarks on integrability and universality.

2. Equilibrium and nonequilibrium states

The most fundamental concept of statistical physics is the distinction between *microstates* and *macrostates* in the description of systems with many degrees of freedom. To fix ideas, consider a classical N-particle system (say, a gas in a box) described by a Hamilton function $H(q, p)$ of position variables $q = (q_1, \ldots, q_{dN})$ and momenta $p = (p_1, \ldots, p_{dN})$. Particles move in a region $\Omega \subset \mathbb{R}^d$ of volume $V = |\Omega|$. Then a *microstate* is simply a point (q, p) in phase space, whereas a *macrostate* will be defined for the purposes of these lectures as a measure $P_X(q, p)\,\mathrm{d}q\,\mathrm{d}p$ parameterized by a set of *macroscopic* state variables (in short *macrovariables*) X. Here, $P_X(q, p)$ is a function on phase space and $\mathrm{d}q\,\mathrm{d}p$ denotes the canonical Liouville measure.

Examples of macrovariables are energy, density, temperature or pressure. The macrovariables parametrizing the macrostate P_X could have a dependence on space and time, but to be useful they should be chosen such that they are slowly varying. This singles out

in particular the conserved quantities of the underlying N-particle system as candidates for macrovariables. The mapping from the microstate (q, p) to the macrovariables X is many-to-one, and the measure $P_X(q, p)\,\mathrm{d}q\,\mathrm{d}p$ gives the probability to find the system in a particular set of microstates (q, p) under the constraint that the macroscopic state is described by X. In principle, the time dependence (if any) of $P_X(q, p)$ is induced by the classical Hamiltonian dynamics of the microstate variables (q, p), but in practice well-chosen macrovariables are often found to satisfy autonomous evolution laws, such as the equations of hydrodynamics. The derivation of macroscopic evolution equations from microscopic Hamiltonian dynamics is the goal of *kinetic theory*. A (much simplified) version of this problem will be addressed below in section 3.4.

In this perspective, *equilibrium states* are a subclass of macrostates which are attained at long times by a system that is isolated or in contact with a time-independent, spatially uniform environment. Characteristic properties of equilibrium states are that

- the macrovariables X are time independent and spatially homogeneous, and
- there are no macroscopic currents (e.g. of mass or energy).

The two most important examples of equilibrium states are as follows.

(a) In an *isolated* system the energy E is conserved, the appropriate macrovariables are $X = (E, V, N)$ and the equilibrium state is the measure induced by the Liouville measure on the energy shell $\{(q, p) : H(q, p) = E\}$. This is known in physics as the *microcanonical* measure.

(b) In a system at *constant temperature* T particles exchange energy with the walls of the box Ω in such a way that the mean energy is fixed. The appropriate macrovariables are then $X = (T, V, N)$ and the equilibrium state is of the form

$$P_{T,V,N} \sim \exp[-\beta H], \quad \beta = 1/T,$$

known as the *canonical* measure.

Having roughly characterized equilibrium states, we may say that *nonequilibrium* states arise whenever the conditions for the establishment of equilibrium are not fulfilled. As such, this definition is about as useful as it would be to define some area of biology as the study of non-elephants. We can be somewhat more precise by making a distinction between the following.

(i) *Systems approaching equilibrium.* By definition, the macrostate of such a system is time dependent. In addition, systems in this class often become spatially inhomogeneous; an important and much studied case are systems undergoing phase separation [26].

(ii) *Nonequilibrium stationary states (NESS).* These systems are kept out of equilibrium by external influences. They are stationary, in the sense that macroscopic state variables are time independent, and they may or may not be spatially homogeneous. In any case they are characterized by non-vanishing macroscopic currents.

Examples for NESS are as follows.

- *Heat conduction.* In a system with boundaries held at different temperatures there is a stationary energy current proportional to the temperature gradient (*Fourier's law*).
- *Diffusion.* In a system coupled to particle reservoirs held at different densities there is a mass current proportional to the density gradient (*Fick's law*).
- *Electric conduction.* Here particles are charged and move under the influence of a constant electric field. The particle current is proportional to the field strength (*Ohm's law*).

Among these three examples, the first two can be further characterized as *boundary driven*, in the sense that the NESS is maintained by boundary conditions on the quantity that is being transported (heat, mass), whereas the last example illustrates a *bulk-driven* NESS maintained by an external field acting in the bulk of the system.

NESS are the simplest examples of nonequilibrium states. Nevertheless, their description in the framework of classical Hamiltonian mechanics is conceptually subtle and technically demanding (see, e.g., [141]). The main reason is that a Hamiltonian system under constant driving inevitably accumulates energy. In order to allow for the establishment of a steady state, dissipation has to be introduced through the coupling to an external reservoir, that is, a system with an infinite number of degrees of freedom.

These difficulties can be avoided by starting from *stochastic* microscopic dynamics. While less realistic on the microscopic level, stochastic models provide a versatile framework for addressing fundamental questions associated with the behavior of many-particle systems far from equilibrium. The class of models of interest here are known in the probabilistic community as *interacting particle systems*. These are lattice models with a discrete (finite or infinite) set of states associated with each lattice site and local interactions. We focus specifically on exclusion processes, which are introduced in the next section.

It is worth pointing out that the notion of equilibrium states in statistical physics, as outlined above, is much more restrictive than the usage of the corresponding term in most areas of mathematics, where an *equilibrium* is commonly understood to be any time-independent solution of some deterministic or stochastic time evolution. Thus, NESS are equilibria in the mathematical sense. In section 3.2 we will give a precise definition of what distinguishes physical equilibria from other time-independent states in the context of continuous time Markov chains.

3. An introduction to exclusion processes

3.1. Definition

The simple exclusion process was introduced in 1970 by Frank Spitzer [122]. Particles occupy the sites of a d-dimensional lattice, which for the purposes of this discussion will be taken to be a finite subset $\Omega \subset \mathbb{Z}^d$. The particles are indistinguishable, which implies that a microstate or configuration of the system is given by

$$\eta = \{\eta_x\}_{x\in\Omega} \in \{0, 1\}^{\Omega},$$

where $\eta_x = 0$ (1) if site x is vacant (occupied). The dynamics can be informally described as follows (for a detailed construction, see [85, 122]).

- Each particle carries a clock which rings according to a Poisson process with unit rate (i.e. the waiting times between rings are exponentially distributed).
- When the clock rings the particle at site x selects a target site y with probability $q_{xy}(\eta)$ and attempts to jump there.
- The jump is performed if the target site is vacant and discarded otherwise; this step implements the *exclusion interaction* between particles and enforces the single occupancy constraint $\eta_x = 0$ or 1.

Together these rules define the exclusion process as a continuous time Markov chain on a finite state space; some general properties of such chains will be discussed in the next section. *Interactions* (beyond the exclusion interaction) can be introduced through the dependence of the jump matrix q_{xy} on the configuration η. Similarly, *inhomogeneity* associated with sites or

particles can be introduced by letting the waiting times and the jump matrix depend explicitly on the particle positions or the particle labels, see [73].

We next restrict the discussion to the one-dimensional case with nearest-neighbor hopping and without inhomogeneities or explicit interactions. Then,

$$q_{xy} = q\delta_{y,x+1} + (1-q)\delta_{y,x-1}.$$

Informally, the particle attempts to jump to the right with probability q and to the left with probability $1-q$. The following cases are of interest.

(i) $q = 1/2$ defines the *symmetric simple exclusion process* (SSEP). We will see below that this is really an equilibrium system. However, when defined on a finite lattice of sites $x = 1, \ldots, L$ and supplemented with boundary rates $\alpha, \beta, \gamma, \delta$ which govern the injection (α, δ) and extraction (γ, β) of particles at the boundary sites $i = 1$ and $i = L$, this model provides a nontrivial example for a boundary-driven NESS [38].

(ii) $q \neq 1/2$ defines the *asymmetric simple exclusion process* (ASEP). When considered on the one-dimensional ring (a lattice with *periodic boundary conditions*) the system attains a bulk-driven NESS in which there is a non-vanishing stationary mass current. This is the simplest realization of a *driven diffusive system* [116].

Note that the boundary conditions are crucial here. On a finite lattice with closed ends, which prevent particles from entering or leaving the system, an *equilibrium* state is established in which the bias in the jump probability is compensated by a density gradient; this is the discrete analog of a gas in a gravitational field, as described by the barometric formula. Another possibility is to consider a finite lattice with open ends at which particles are injected and extracted at specified rates [71]. This leads to a NESS with a surprisingly complex structure, see [20] for a review.

(iii) $q = 1$ (or 0) defines the *totally asymmetric simple exclusion process* (TASEP). In contrast to the case of general q, this process can also be formulated in discrete time [143]. In one time step $t \to t+1$, all particles attempt to move to the right (say) simultaneously and independently with probability $\pi \in (0, 1]$; moves to vacant sites are accepted and moves to occupied sites discarded. Such a discrete time dynamics cannot be defined for $0 < q < 1$, because it would lead to conflicts when different particles attempt to simultaneously access the same vacant site.

For $\pi \to 0$, the discrete time TASEP (*dTASEP*) reduces to the continuous time process in rescaled time πt, while for $\pi = 1$ it becomes a deterministic cellular automaton which has number 184 in Wolfram's classification [77, 142]. The case of general π has been studied mostly in the context of vehicular traffic modeling [29, 117].

Note that in terms of the waiting time picture sketched above, the discrete time dynamics corresponds to replacing the exponential waiting time distribution by a geometric distribution with support on integer times only. The exponential and geometric waiting time distributions are the only ones that encode *Markovian* dynamics [75]. The waiting time representation will play an important role in the exact solution of the dTASEP presented below in section 6.

3.2. Continuous time Markov chains

Before discussing some specific properties of exclusion processes, we outline the general setting of continuous time Markov chains (see [106] for an introduction). Consider a Markov chain with a finite number of states $i = 1, \ldots, C$ and transition rates Γ_{ij}. The rates define the dynamics in the following way.

- When the chain is in state i at time t, a transition to state $j \neq i$ occurs in the time interval $[t, t + \mathrm{d}t]$ with probability $\Gamma_{ij}\mathrm{d}t$.

The key quantity of interest is the transition probability

$$P_{ki}(t) = \text{Prob}[\text{state } i \text{ at } t | \text{state } k \text{ at } 0] \equiv P_i(t),$$

where the initial state k is included through the initial condition $P_i(0) = \delta_{ik}$. The transition probability satisfies the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_i = \sum_{j\neq i}\Gamma_{ji}P_j - \sum_{j\neq i}\Gamma_{ij}P_i = \sum_j A_{ji}P_j, \tag{1}$$

which is known as the *master equation* in physics [138] and as the *forward equation* in the theory of stochastic processes [106]. Here the *generator matrix*

$$A_{ij} = \begin{cases} \Gamma_{ij} & : \quad i \neq j \\ -\sum_{k\neq i}\Gamma_{ik} & : \quad i = j \end{cases}$$

has been introduced. The master equation simply accounts for the balance of probability currents going in and out of each state of the Markov chain. To bring out this structure we rewrite (1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}P_i = \sum_j K_{ij}, \qquad K_{ij} = \Gamma_{ji}P_j - \Gamma_{ij}P_i, \tag{2}$$

where K_{ij} is the *net probability current* between states i and j [144]. If the chain is *irreducible*, in the sense that every state can be reached from every other state through a connected path of nonzero transition rates, the solution of (1) approaches at long times a unique, stationary invariant measure P_i^* determined by the condition

$$\sum_j A_{ji}P_j^* = 0. \tag{3}$$

The invariant measure is the left eigenvector of the generator matrix, with eigenvalue zero. Based on (2) we can rewrite (3) as

$$\sum_j K_{ij}^* = 0 \qquad \text{with} \qquad K_{ij}^* = \Gamma_{ji}P_j^* - \Gamma_{ij}P_i^*. \tag{4}$$

Two classes of Markov chains may now be distinguished depending on how the stationarity condition (4) is realized.

(i) $K_{ij}^* = 0 \; \forall \; i, j$. In this case the probability currents cancel between any two states i, j:

$$\Gamma_{ij}P_i^* = \Gamma_{ji}P_j^*, \tag{5}$$

a condition that is known in physics as *detailed balance*. In the mathematical literature, Markov chains with this property are called *reversible*, because (5) implies that the weight of any trajectory (with respect to the invariant measure) is equal to that of its image under time reversal [65, 106]. Detailed balance or, equivalently, reversibility is a fundamental property that any stochastic model of a physical system *in equilibrium* must satisfy, because equilibrium states are distinguished by invariance under time reversal.

(ii) $K_{ij}^* \neq 0$ at least for some pairs of states i, j. Such a Markov chain is irreversible and describes a system in a NESS.

Examples for both kinds of situations will be encountered in the next section.

3.3. Stationary measure of the exclusion process

We consider the ASEP on a ring of L sites with a fixed number N of particles. The total number of microstates η is then $C = \binom{L}{N}$ and the transition rates are

$$\Gamma(\eta \to \eta') = \begin{cases} q & : \quad (\ldots \bullet \circ \ldots) \to (\ldots \circ \bullet \ldots) \\ 1-q & : \quad (\ldots \circ \bullet \ldots) \to (\ldots \bullet \circ \ldots) \\ 0 & : \quad \text{else.} \end{cases} \tag{6}$$

Here $(\ldots \bullet \circ \ldots)$ denotes a local configuration with an occupied site ($\bullet$) to the left of a vacant site ($\circ$), and it is understood that only configurations η, η' that differ by the exchange of a single particle-vacancy pair are connected through nonzero transition rates. The stationary measure $P^*(\eta)$ is determined by the condition

$$\sum_{\eta'} \Gamma(\eta' \to \eta) P^*(\eta') = \sum_{\eta'} \Gamma(\eta \to \eta') P^*(\eta) \quad \forall\, \eta. \tag{7}$$

As the simplest possibility, let us assume that the invariant measure is uniform on the state space:

$$P^*(\eta) = \binom{L}{N}^{-1} \quad \Rightarrow \quad K^*(\eta, \eta') = [\Gamma(\eta' \to \eta) - \Gamma(\eta \to \eta')]\binom{L}{N}^{-1}. \tag{8}$$

We discuss separately the symmetric and the asymmetric process.

- $q = 1/2$ (SSEP). Here the rate $q = 1 - q = 1/2$ for all allowed processes, and for each allowed process the reverse process occurs at the same rate. We conclude that the detailed balance condition (5) holds in this case, $K^* = 0$, and the SSEP is reversible as announced previously.
- $q \neq 1/2$ (ASEP). Because for any allowed process with rate q the reverse process occurs at rate $1 - q \neq q$ and vice versa, detailed balance is manifestly broken, $K^* \neq 0$, and we are dealing with an irreversible NESS. However, we now show that the uniform measure (8) is nevertheless invariant. To see this, consider the total transition rates for all processes leading into or out of a given configuration η. We have

  $$\Gamma^{\text{in}}_{\text{tot}}(\eta) = \sum_{\eta'} \Gamma(\eta' \to \eta) = q\mathcal{N}_{\circ\bullet}(\eta) + (1-q)\mathcal{N}_{\bullet\circ}(\eta),$$

  where $\mathcal{N}_{\circ\bullet}(\eta)$ denotes the number of pairs of sites with a particle to the right of a vacancy in the configuration η. Similarly

  $$\Gamma^{\text{out}}_{\text{tot}}(\eta) = \sum_{\eta'} \Gamma(\eta \to \eta') = q\mathcal{N}_{\bullet\circ}(\eta) + (1-q)\mathcal{N}_{\circ\bullet}(\eta).$$

  A little thought reveals that $\mathcal{N}_{\bullet\circ}(\eta) = \mathcal{N}_{\circ\bullet}(\eta)$ for any configuration η. Hence $\Gamma^{\text{in}}_{\text{tot}}(\eta) = \Gamma^{\text{out}}_{\text{tot}}(\eta)$ for any q, and the stationarity condition (7) is satisfied for the uniform measure (8).

  A few remarks are in order.

(i) The invariance of the uniform measure (8), and the fact that it is independent of the bias q, relies crucially on the ring geometry. With open boundary conditions allowing for the injection and extraction of particles, both the SSEP and the ASEP display nontrivial invariant measures characterized by long-ranged correlations and the possibility of phase transitions [20, 38]. For example, for the SSEP with boundary densities ρ_L at $x = 1$ and ρ_R at $x = L$, one finds a linear mean density profile, as expected from Fick's law, but in

addition there are long-ranged density–density correlations on the scale L, which take the form [38, 123]:

$$\mathbb{E}(\eta_{L\xi}\eta_{L\xi'}) - \mathbb{E}(\eta_{L\xi})\mathbb{E}(\eta_{L\xi'}) = -\frac{\xi(1-\xi')}{L}(\rho_L - \rho_R)^2.$$

Here $\xi, \xi' \in [0, 1]$ are scaled position variables with $\xi < \xi'$.

(ii) The invariant measure of the dTASEP on the ring is *not* uniform. Rather, one finds a Gibbs measure with repulsive nearest-neighbor interactions between the particles [115, 117, 143]. This means that the probability of a configuration can be written as a product of pair probabilities:

$$P^*_\rho(\eta) \sim \prod_x p_\rho(\eta_x, \eta_{x+1}), \tag{9}$$

where the limit $N, L \to \infty$ at fixed density

$$\rho = \mathbb{E}(\eta_x) = N/L \tag{10}$$

is implied and

$$p_\rho(0, 1) = p_\rho(1, 0) = \frac{1 - \sqrt{1 - 4\pi\rho(1-\rho)}}{2\pi}, \tag{11}$$

$$p_\rho(0, 0) = 1 - \rho - p_\rho(1, 0), \qquad p_\rho(1, 1) = \rho - p_\rho(1, 0). \tag{12}$$

For $\pi \to 0$ this reduces to a Bernoulli measure of independent particles (see section 3.4), whereas for $\pi \to 1$ we have $p_\rho(1, 0) \to (1-|1-2\rho|)/2$, which implies that $p_\rho(1, 1) \to 0$ for $\rho < 1/2$ and $p_\rho(0, 0) \to 0$ for $\rho > 1/2$. At $\pi = 1$ and mean density $\rho = 1/2$, the measure is concentrated on the two configurations $\eta_x = [1 \pm (-1)^x]/2$.

(iii) The invariance of the uniform measure for the ASEP on the ring is an example of *pairwise balance* [119], a property that generalizes the detailed balance condition (5) into the form

$$\Gamma(\eta \to \eta')P^*(\eta) = \Gamma(\eta'' \to \eta)P^*(\eta'').$$

This means that for each configuration η' contributing to the outflux of probability out of the state η there is a configuration η'' whose influx contribution precisely cancels that outflux. In other words, the terms in the sums on the two sides of (7) cancel pairwise.

3.4. Hydrodynamics

An important goal in the study of stochastic interacting particle systems is to understand how deterministic evolution equations emerge from the stochastic microscopic dynamics on large scales [18, 66, 124]. This is similar to the (much harder) problem of deriving hydrodynamic equations from the Newtonian dynamics of molecules in a gas or a fluid. The mathematical procedure involved in the derivation of macroscopic evolution equations for systems with conserved quantities is therefore referred to as the *hydrodynamic limit*. Here we give a heuristic sketch of hydrodynamics for the ASEP.

The key input going into the hydrodynamic theory is the relationship between the particle density ρ and the stationary particle current J. The particle current is defined as the net number of particles jumping from a site x to the neighboring site $x + 1$ per unit time, which is independent of x in the stationary state. From the definition of the ASEP we have

$$J = q\mathbb{E}[\eta_x(1 - \eta_{x+1})] - (1 - q)\mathbb{E}[\eta_{x+1}(1 - \eta_x)],$$

where expectations are taken with respect to the invariant measure. Since all configurations of N particles on the lattice of L sites are equally probable,

$$\mathbb{E}[\eta_x(1-\eta_{x+1})] = \mathbb{E}[\eta_{x+1}(1-\eta_x)] = \frac{N}{L}\frac{(L-N)}{L-1}.$$

This is just the probability of finding a filled site next to a vacant site, which is obtained by first placing one out of N particles in one of L sites, and then placing one out of $L-N$ vacancies in one of the remaining $L-1$ sites. We conclude that

$$J = \frac{(2q-1)\rho(1-\rho)}{1-1/L} \to (2q-1)\rho(1-\rho) \qquad \text{for} \quad L \to \infty,$$

where the particle density (10) is kept fixed. Similarly

$$\mathbb{E}[\eta_x\eta_y] = \frac{N}{L}\frac{N-1}{L-1} \to \rho^2 = \mathbb{E}[\eta_x]\mathbb{E}[\eta_y] \qquad \text{for} \quad L \to \infty \tag{13}$$

for any pair of sites $x \neq y$. This implies that in the invariant measure on the ring, restricted to a fixed finite number of sites, for $L \to \infty$ each site is occupied independently with probability ρ (the Bernoulli measure).

We can now formulate the basic idea of the hydrodynamic limit [83]. Suppose we start the ASEP at time $t = 0$ from a Bernoulli measure with a slowly varying density $\rho(x, 0)$. Here 'slowly varying' means that variations occur on a scale $\ell \gg 1$. Since the invariant measure of the ASEP is a Bernoulli measure of *constant* density, it is plausible that if ℓ is chosen large enough, the evolving measure will remain close to a Bernoulli measure with a time- and space-dependent density $\rho(x, t)$ at all times, and because the particle density is locally conserved, the evolution equation for $\rho(x, t)$ must be of conservation type:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}j(x,t) = 0. \tag{14}$$

In the limit $\ell \to \infty$ we may expect, in the spirit of a law of large numbers, that the local particle current $j(x, t)$ converges to the stationary current associated with the local density $\rho(x, t)$:

$$j(x,t) \to J(\rho(x,t)),$$

such that (14) becomes an autonomous, deterministic hyperbolic conservation law

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}J(\rho) = 0 \tag{15}$$

for the density profile $\rho(x, t)$. Equation (15) is known as the *Euler* equation for the ASEP, because similarly to the Euler equation in fluid mechanics it lacks a second-order 'viscosity' term $\nu\partial^2\rho/\partial x^2$. It must be emphasized that such a term does *not* appear to leading order, when the hydrodynamic limit is carried out at fixed $q \neq 1/2$. It is only present in the *weakly asymmetric* case, which implies that $q \to 1/2$ in the limit $\ell \to \infty$ such that $\ell(q - 1/2)$ is kept fixed [37].

The Euler equation (15) has been rigorously established for a wide range of models, including cases in which the invariant measure and the current–density relation $J(\rho)$ are not explicitly known [120]. We conclude this section by a brief discussion of the properties of the nonlinear PDE (15), assuming a general (but convex) current–density relation with $J(0) = J(1) = 0$. This includes in particular the dTASEP for which

$$J(\rho) = \pi p_\rho(1,0) = \tfrac{1}{2}[1 - \sqrt{1 - 4\pi\rho(1-\rho)}]. \tag{16}$$

(i) *Shock formation*. Hyperbolic conservation laws of the form (15) can generally be solved by the *method of characteristics*. To this end we first rewrite (15) in the form

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{17}$$

where

$$c(\rho) = \frac{dJ}{d\rho}. \tag{18}$$

A characteristic is a trajectory of a point of constant density, and the key observation is that the characteristics of (17) are straight lines. Denoting by $v_{\rho_0}(x,t)$ the position of a point of density $\rho_0 = \rho(x,0)$ at time t, we have to satisfy the condition

$$\rho(v_{\rho_0}(x,t),t) = \rho_0 = \rho(x,0)$$

at all times. Taking the time derivative of this relation and using (17) we see that the solution is

$$v_{\rho_0}(x,t) = x + c(\rho_0)t,$$

i.e. points of constant density travel at the *kinematic wave speed* (18).

The convexity of the current–density relation implies that $c(\rho)$ is a decreasing function of the density. As a consequence characteristics collide in regions of increasing initial density, $\mathrm{d}\rho(x,0)/\mathrm{d}x > 0$, leading to the formation of density discontinuities (*shocks*) in finite time. At this point the description by the PDE (15) breaks down, but the speed V of a shock separating regions of density ρ_L on the left and $\rho_R > \rho_L$ on the right is easily inferred from mass conservation to be given by

$$V = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L}. \tag{19}$$

Note that $V \to c$ for $\rho_L \to \rho_R$. On the microscopic level shocks are represented by the *shock measures* of the ASEP [45, 47]. These are inhomogeneous invariant measures on $\mathbb{Z}$ which approach Bernoulli measures with density ρ_L and ρ_R for $x \to -\infty$ and $x \to \infty$, respectively. The microscopic structure of shocks has been studied in considerable detail [39].

(ii) *Rarefaction waves.* If the initial density profile is a step function

$$\rho(x,0) = \begin{cases} \rho_L & : \quad x < 0 \\ \rho_R & : \quad x > 0 \end{cases} \tag{20}$$

with $\rho_L > \rho_R$, a diverging fan of characteristics forms leading to a broadening, self-similar density profile

$$\rho(x,t) = \begin{cases} \rho_L & : \quad x < c(\rho_L)t \\ \rho_R & : \quad x > c(\rho_R)t \\ \phi(x/t) & : \quad c(\rho_L) < x/t < c(\rho_R), \end{cases} \tag{21}$$

where the shape function $\phi(\xi)$ can be computed from the current–density relation $J(\rho)$. Inserting the ansatz (21) into (17) we see that

$$\phi(\xi) = c^{-1}(\xi). \tag{22}$$

For the continuous time ASEP the interpolating shape is linear:

$$\phi(\xi) = \frac{1}{2}\left(1 - \frac{\xi}{2q-1}\right).$$

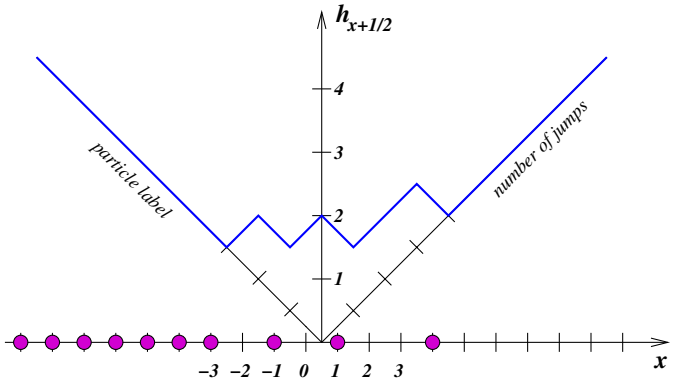

Figure 1. Schematic of the mapping between a configuration of the ASEP and the corresponding height configuration $h_{x+\frac{1}{2}}$ (bold line). Initially all sites $x \leqslant 0$ in the ASEP are occupied, and all sites $x > 0$ are vacant (step initial condition). The ASEP occupation variables determine the height differences according to (23). At the same time, the height increment $h_{x+\frac{1}{2}}(t) - h_{x+\frac{1}{2}}(0)$ counts the net number of particles that have crossed the bond $(x, x+1)$ from left to right up to time t. In the figure two particles have crossed the origin, and $h_{\frac{1}{2}} = 2$. Rotating the height configuration by 45° provides a representation of the net number of jumps a given particle has undergone as a function of the particle label, counted backward from $x = 0$ (the lth particle is the one that was located at $x = -l$ at time $t = 0$).

3.5. Mapping to a growth model

The representation of the one-dimensional ASEP as a growth model seems to have been formulated first by Rost [107]. It is illustrated in figure 1 for a step initial condition, where all sites to the left of the origin are occupied ($\eta_x = 1$ for $x \leqslant 0$) and all sites to the right of the origin are empty ($\eta_x = 0$ for $x > 0$). This initial condition will also play a central role in section 5.

The mapping assigns to every configuration $\eta = \{\eta_x\}$ of the ASEP a configuration of height variables $\{h_{x+\frac{1}{2}}\}$, where the shift of the index indicates that the height variable $h_{x+\frac{1}{2}}$ lives on the bond connecting the ASEP sites x and $x+1$. After fixing the height at a reference point, e.g. by setting $h_{\frac{1}{2}} = 0$, the height configuration is uniquely determined by the relation

$$h_{x+\frac{1}{2}} - h_{x-\frac{1}{2}} = \tfrac{1}{2} - \eta_x. \tag{23}$$

The ASEP occupation variable encodes the local slopes of the height profile, which takes the values $h_{x+\frac{1}{2}} - h_{x-\frac{1}{2}} = \pm\frac{1}{2}$, hence the name 'single step model' in the physics literature [92, 98].

The step initial condition corresponds to the initial height profile

$$h_{x+\frac{1}{2}}(0) = \tfrac{1}{2}|x|,$$

which accounts for the designation as a 'corner growth model' in the mathematical literature. It can be seen from figure 1 that a particle jumping across a bond to the right (left) increases (decreases) the corresponding height variable by one unit. Thus, the height $h_{x+\frac{1}{2}}$ is an odd (even) multiple of $\frac{1}{2}$ for odd (even) values of x, and the height change $h_{x+\frac{1}{2}}(t) - h_{x+\frac{1}{2}}(0)$ is equal to the net number of particles that have jumped across the bond $(x, x+1)$ from left to right up to time t (jumps from right to left are counted with a negative sign). Finally, for the special case of the step initial condition, the net number of jumps (forward minus backward) performed by a given particle can also be read off from the height profile (see figure 1).

The mapping (23) is clearly not restricted to the step initial condition. Of particular interest are translationally invariant initial conditions, which can be constructed deterministically or stochastically. For example, to generate a deterministic initial condition of density $\rho = 1/n$, one simply places a particle at every nth site of the lattice, and a stochastic initial condition is obtained by occupying sites independently with probability ρ. The two types of initial conditions differ in the *roughness* of the corresponding height configuration, which is quantified by the *height difference correlation function*

$$G(r) = \mathbb{E}[(h_{y+r} - h_y)^2] - \mathbb{E}[h_{y+r} - h_y]^2. \tag{24}$$

An ensemble of height configurations on $\mathbb{Z}$ is said to be *smooth* if $\lim_{r\to\infty} G(r) < \infty$ and *rough* otherwise [79]. The deterministic initial conditions described above are smooth in this sense, whereas for the stochastic initial condition a simple computation using (23) and (13) shows that

$$G(r) = \rho(1-\rho)|r|. \tag{25}$$

4. The KPZ conjecture

The asymmetric exclusion process and the equivalent growth model introduced in the preceding subsection are representatives of a large class of models, which was brought to the forefront of research in nonequilibrium statistical physics in 1986 by a seminal paper of Kardar, Parisi and Zhang (KPZ) [63]. Working in the framework of a phenomenological stochastic continuum description, they formulated what may be called a *universality hypothesis* encompassing the fluctuation properties of a large class of different microscopic models[3]. The classic period of research in this area has been extensively reviewed in the literature [55, 72, 79]. Here we aim to give a concise and simple presentation of the KPZ conjecture, in order to place the more recent developments (to be elaborated in the following sections) into their proper context.

4.1. The Kardar–Parisi–Zhang equation

We start from the hydrodynamic equation (15) with a general current–density relation $J(\rho)$. Since we are interested in fluctuations around a state of constant mean density $\bar\rho$, we write $\rho(x,t) = \bar\rho + u(x,t)$ and expand to second order in u, which yields

$$\frac{\partial u}{\partial t} = -c(\bar\rho)\frac{\partial u}{\partial x} - \lambda u\frac{\partial u}{\partial x}, \tag{26}$$

where

$$\lambda = \frac{\mathrm{d}^2 J}{\mathrm{d}^2\rho}(\bar\rho). \tag{27}$$

The linear drift term on the right-hand side can be eliminated by a Galilei transformation $x \to x - ct$, which leaves us with what is known (for $\lambda = 1$) as the *inviscid Burger equation*.

Now fluctuations are introduced (in the spirit of fluctuating hydrodynamics [124]) by adding a random force to the right-hand side of (26). In order to guarantee mass conservation, this term must take the form of a derivative $-\partial\zeta/\partial x$ of a stochastic process $\zeta(x,t)$ in space and time. This is assumed to be a stationary Gaussian process with zero mean and a covariance function

$$\mathbb{E}[\zeta(x,t)\zeta(x',t')] = a_x^{-1}a_t^{-1}G[(x-x')/a_x, (t-t')/a_t] \tag{28}$$

[3] For an introduction to the idea of universality from a mathematical perspective, see [34].

which vanishes beyond a small correlation length a_x and a short correlation time a_t. Usually one takes formally[4] $a_x, a_t \to 0$, which reduces the right-hand side of (28) to a product of δ-functions

$$\mathbb{E}[\zeta(x,t)\zeta(x',t')] \to D\delta(x-x')\delta(t-t') \tag{29}$$

and turns the process $\zeta(x,t)$ into spatio-temporal white noise of strength D. This rather violent driving has to be compensated by a small viscosity term $\nu\partial^2 u/\partial x^2$ with $\nu > 0$. Putting all ingredients together we thus arrive at the *stochastic Burgers equation*

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} - \lambda u\frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} \equiv -\frac{\partial}{\partial x} j(x,t), \tag{30}$$

first introduced in the context of randomly stirred fluids [51] and subsequently applied to fluctuations in the exclusion process by van Beijeren, Kutner and Spohn [137].

To establish the connection to growth models we proceed in analogy to the discrete case discussed in section 3.5. We introduce the *height function* $h(x,t)$ through the time-integrated particle current,

$$h(x,t) = \int_0^t j(x,s)\,\mathrm{d}s. \tag{31}$$

Supplementing this with the initial condition $u(x,0) = 0$, it follows from the conservation law for u that

$$\frac{\partial h}{\partial x} = -u, \tag{32}$$

and therefore

$$\frac{\partial h}{\partial t} = \nu\frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 - \zeta, \tag{33}$$

which is precisely the KPZ equation [63]. In general there is also a constant term on the right-hand side of (33) which has been set to zero. The defining feature of the equation is the quadratic nonlinearity on the right-hand side, which is present whenever $\lambda \neq 0$, that is, when the current is a (generic) nonlinear function of the density[5] (compare to (27)).

It is important to clearly understand the relation between the stochastic PDEs (30), (33) and the underlying discrete particle systems. The coefficient λ in (33) is defined through the current–density relation according to (27), but the viscosity ν and the noise strength D in (29) do not directly appear on the discrete level. To give these coefficients a consistent interpretation, we start from the observation [51, 57] that the invariant measure of (30) with spatio-temporal white-noise driving is spatial white noise[6] with strength $D/2\nu$. This is easy to check for the linearized equation ($\lambda = 0$) but it remains true also for $\lambda \neq 0$, somewhat analogous to the invariance of the uniform measure for the ASEP discussed in section 3.3. As a consequence, the spatial statistics of $h(x,t)$ for long times is that of a Wiener process with 'diffusion constant' $D/4\nu$ *in space*[7]:

$$\lim_{t\to\infty} \mathbb{E}[(h(x,t) - h(x',t))^2] = \frac{D}{2\nu}|x-x'| \equiv A|x-x'|. \tag{34}$$

[4] In the hydrodynamic context [51] it is also of interest to consider the solutions of (30) on scales *small* compared to the spatial driving scale a_x.

[5] Note that it is possible to have $\lambda \neq 0$ even if the current itself vanishes at the specific mean density under consideration (see [23] for an example).

[6] This remains true for certain discretizations of (33) [79, 111].

[7] An immediate consequence of (34) is that typical configurations of $h(x,t)$ are non-differentiable. This is the origin of the ill-posedness of the KPZ equation.

This relation holds also on the discrete level, provided $|x - x'|$ is large compared to the correlation length of the particle system, and it identifies the ratio $A = D/2\nu$ as a property of the invariant measure of the latter; for the continuous time ASEP we read off the relation $A = \bar{\rho}(1 - \bar{\rho})$ from (25), and for the discrete time TASEP A can be computed from the transition probabilities (11), (12) [see (40)]. It can be seen from relation (32) (or its discrete analog (24)) that the height difference correlation function is a measure of the fluctuations in the particle number in the interval between x and x'. For this reason A has been referred to as a (nonequilibrium) compressibility [54].

We note for later reference that the KPZ equation (33) can be linearized using the Hopf–Cole transformation

$$Z(x,t) = \exp\left[-\frac{\lambda}{2\nu}h(x,t)\right], \tag{35}$$

which was originally applied to the deterministic Burgers equation [30, 56] and rediscovered in the context of (33) by Huse, Henley and Fisher [57]. Indeed, using (33) we see that $Z(x, t)$ evolves according to a heat equation with a multiplicative stochastic force

$$\frac{\partial Z}{\partial t} = \nu\nabla^2 Z + \frac{\lambda}{2\nu}\zeta(x,t)Z(x,t). \tag{36}$$

The formal solution of (36) is a Wiener path integral describing the weight $Z(x, t)$ of a Brownian path (or 'directed polymer' [63]) subject to the random spacetime potential $\frac{\lambda}{2\nu}\zeta(x,t)$.

4.2. The universality hypothesis

The considerations in the preceding subsection suggest that the details of the underlying particle system enter the large-scale fluctuations properties only through the two parameters λ and A. These parameters define characteristic scales of height, length and time, which can be used to non-dimensionalize any correlation function of interest. In the non-dimensional variables the correlation functions are then conjectured to be *universal*, i.e. independent of the specific microscopic model. This is the essence of the universality hypothesis.

As an illustration, consider the probability distribution of the height $h(x, t)$ at a given point x, corresponding to the time-integrated current through a fixed bond in the exclusion process. Because of translational invariance, this cannot depend on x, and we have to find a combination of λ, A and t that has the dimension of h. Denoting the dimension of a quantity X by $[X]$, we read off from (33) that

$$[\lambda] = \frac{[x]^2}{[h][t]}$$

and from (34) that

$$[A] = \frac{[h]^2}{[x]}.$$

The unique combination with the dimension $[h]$ is $(A^2|\lambda|t)^{1/3}$, and hence we expect that the rescaled height fluctuation

$$\tilde{h} = \frac{h}{(A^2|\lambda|t)^{1/3}} \tag{37}$$

should have a universal distribution. For example, the variance of the height is predicted to be of the form [76]

$$\mathbb{E}[h(x,t)^2] - (\mathbb{E}[h(x,t)])^2 = c_2(A^2|\lambda|t)^{2/3} \tag{38}$$

with a universal constant c_2 which is independent of the specific model or of model parameters such as the update probability π in the dTASEP (see remark 5.5). Similarly, the unique combination of λ, A and t that has the dimension $[x]$ of length is

$$\ell(t) = (A\lambda^2)^{1/3} t^{2/3}, \tag{39}$$

which defines the *correlation length* of fluctuations; note that correlations spread *superdiffusively*, that is, faster than $t^{1/2}$ [137]. This is in contrast to the case $\lambda = 0$, where a straightforward solution of (33) shows that the correlation length grows diffusively, and height fluctuations are Gaussian and of order $t^{1/4}$ [72]. This behavior has been explicitly demonstrated for interacting particle systems in which the current is a linear (or constant) function of the density[8] [14, 46, 74].

As an illustration of these considerations, and for later reference, we compute the scale factor $A^2|\lambda|$ for the dTASEP at density $\bar{\rho} = 1/2$. Taking two derivatives of the current function (16) we find

$$\lambda_{\mathrm{dTASEP}}(1/2) = -\frac{2\pi}{\sqrt{1-\pi}}.$$

To determine the compressibility A we appeal to the equivalence of the invariant measure (9) to the equilibrium state of the one-dimensional Ising chain[9]. Ising spins σ_i are canonically related to the occupation variables η_i by $\sigma_i = 1 - 2\eta_i = \pm 1$. The transition probabilities (11), (12) make up the *transfer matrix* of the Ising chain, with the density ρ playing the role of the magnetic field (which vanishes when $\rho = 1/2$) and the update probability π controlling the nearest-neighbor coupling; since $p_{1/2}(0, 1) > p_{1/2}(1, 1)$ the coupling is *antiferromagnetic* for $\pi > 0$. Particle number fluctuations translate to fluctuations of the magnetization, and hence the compressibility is proportional to the magnetic susceptibility of the Ising chain. This can be computed from the free energy per spin, which is proportional to the logarithm of the largest eigenvalue of the transfer matrix, by taking two derivatives with respect to the magnetic field. The final result is

$$A_{\mathrm{dTASEP}}(1/2) = \frac{1}{4}\frac{p_{1/2}(1,1)}{p_{1/2}(0,1)} = \frac{1}{4}\sqrt{1-\pi}, \tag{40}$$

and we conclude that

$$(A^2|\lambda|)_{\mathrm{dTASEP}} = \tfrac{1}{8}\pi\sqrt{1-\pi}. \tag{41}$$

The early work on KPZ-type processes was mostly concerned with establishing the universality of the $t^{2/3}$-scaling of the variance (38) which, once the role of A and λ has been recognized, is essentially a consequence of dimensional analysis [72]. Numerical evidence of universality in a more refined sense, which encompasses universal *amplitudes* like c_2 in (38), was presented in [76], where it was also pointed out that different *universality classes* characterized by the same $t^{2/3}$ scaling but different amplitudes may arise from different initial and boundary conditions. Specifically, three cases were identified.

(I) *Growth from a flat surface without fluctuations*. In the language of exclusion processes, this corresponds to a *deterministic* initial condition of constant density; for example, the case $\bar{\rho} = 1/2$ is realized by occupying all odd or all even sites of the lattice.

[8] It is also possible to construct situations where the leading order nonlinearity in the expansion of the current is of cubic or higher order in the density fluctuations, so that $\lambda = 0$ but the problem remains nonlinear [40]. Non-rigorous analysis indicates that such nonlinearities are irrelevant (in the sense that the diffusive behavior is preserved) when of quartic order or higher, while in the cubic case fluctuations spread weakly superdiffusively as $t^{1/2}(\ln t)^{1/4}$ [19, 41].

[9] The one-dimensional Ising chain is treated in most textbooks on statistical physics, see e.g. [97].

(II) *Growth from a flat surface with stationary roughness*, in the sense of (34). This corresponds to starting the exclusion process in a configuration generated from the invariant measure, e.g. a Bernoulli initial condition of density ρ for the continuous time ASEP. In this case the universal fluctuations of interest are visible only if the density is chosen such that the kinematic wave speed $c(\rho) = 0$; otherwise they will be masked by the fluctuations in the initial condition which drift across the observation point. The drift can be eliminated by moving the observation point at the kinematic wave speed $c(\rho)$.

(III) *Growth of a cluster from a seed.* For the exclusion process this corresponds to a step initial condition of the form (20) with $\rho_L > \rho_R$. When $\rho_L > 1/2 > \rho_R$ the relation (22) ensures that the density at the origin $x = 0$ remains at $\phi = c^{-1}(0) = 1/2$ at all times. As in case II, current fluctuations at other values of ρ can be studied by moving the observation point along a general characteristic $x/t = \xi$ with $\phi(\xi) = \rho$.

Early attempts to derive refined universal information, such as amplitudes and scaling functions, directly from the KPZ equation met with limited success [2, 52, 76]. A full understanding of the universality classes of the one-dimensional KPZ equation became available only through the spectacular developments that were triggered a decade ago by the paper of Baik, Deift and Johansson [6]. In the next three sections we explain the key steps of this development along the lines of the work of Johansson [59] and return to the broader issue of KPZ universality in section 8.

5. An exactly solvable model: dTASEP with step initial conditions

In this section we begin our discussion of Johansson's result [59] on the fluctuations of the particle flux for discrete time TASEP (dTASEP) with step initial data. We formulate the result in theorem 5.4 below and we compare it with the predictions of the KPZ theory described in the previous section.

Let us first recall the dTASEP model that has been introduced in section 3.1 (iii). We denote the infinitely many particles of the system by integers $j = 0, 1, 2, \ldots$ and their respective positions at integer times $t = 0, 1, 2, \ldots$ by $x_j(t) \in \mathbb{Z}$. We assume step initial conditions $x_j(0) = -j$. Jumps to the right $x_j(t+1) = x_j(t)+1$ are attempted at every time step $t \geqslant 0$ by all particles $j \geqslant 0$ independently with probability π, but have to be discarded by the exclusion property if at time t the receiving site $x_j(t) + 1$ is occupied by another particle of the system, i.e. if $x_{j-1}(t) = x_j(t) + 1$. In this case, particle j remains on its site, $x_j(t + 1) = x_j(t)$.

Definition 5.1. *We denote by $\mathbb{P}_\pi$ the probability measure on the (total) motion of the particle system that is induced by the stochastic process described above.*

Let us first look at an example and compute the probability that the motion depicted in figure 2 occurs. To do this we only need to count for each particle $j = 0, 1, 2, 3$ how many times it had a choice to jump and how often it actually jumped.

	♯ choices	♯ jumps	♯ stays
$j = 0$	10	5	5
$j = 1$	11	5	6
$j = 2$	10	4	6
$j = 3$	9	4	5
total	40	18	22

By the assumed stochastic independence of all jumps we have

$$\mathbb{P}_\pi \text{ (figure 2 occurs)} = \pi^{18}(1-\pi)^{22}\,.$$

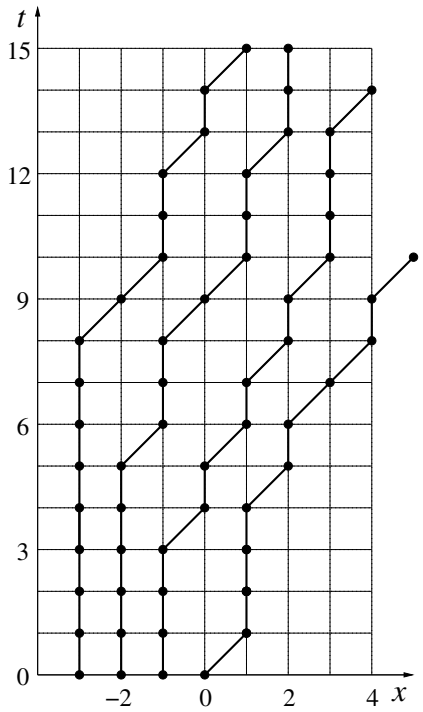

Figure 2. Sample path for dTASEP where only the motion of the four rightmost particles $j = 0, 1, 2, 3$ is displayed up to some time $t \leqslant 15$.

Next we turn to the flux which is the quantity that we want to analyze.

Definition 5.2. *For $r \in \mathbb{Z}$, $t \in \mathbb{N}$ we denote the total flux through the bond between sites r and $r + 1$ up to time t by*

$$F_r(t) := \sharp\{j \in \mathbb{N} : x_j(t) > r\} - \sharp\{j \in \mathbb{N} : x_j(0) > r\},$$

i.e. the total number of particles that have crossed from site r to $r + 1$ during the time interval $[0, t]$.

For example, in the particular situation displayed in figure 2 we have

t	3	6	9	12
$F_{-1}(t)$	0	1	2	2
$F_0(t)$	1	2	2	3
$F_1(t)$	0	1	2	2

From now on we will only consider the flux $F_0(t)$ through the bond between sites 0 and 1 in order to keep the presentation as simple as possible.

Let us first recall what the discussion on the hydrodynamic limit presented in section 3.4 implies for the current at $x = 0$. We are exactly in the situation of the rarefaction wave (see (ii) of section 3.4) with $\rho_L = 1$ and $\rho_R = 0$ and with $J(\rho)$ given by (16). Since $c(1/2) = J'(1/2) = 0$ we learn from (21) and (22) that $\rho(0, t) = \phi(0) = c^{-1}(0) = 1/2$ and again by (16) it follows that the current $j(0, t)$ is given by

$$j(0, t) = J(\rho(0, t)) = \tfrac{1}{2}(1 - \sqrt{1 - \pi}) =: J_\pi. \tag{42}$$

We therefore expect that $F_0(t)$ is approximately given by $J_\pi t$. Indeed, it is a corollary of theorem 5.4 below that $F_0(t)/t$ converges with probability 1 to J_π as $t \to \infty$.

Now we turn to the more detailed predictions of the KPZ theory. As was explained in section 3.5, the flux $F_0(t)$ corresponds to the height $h(1/2, t)$. The dimensional analysis of section 4 suggests that the fluctuations of $F_0(t) - J_\pi t$ are of order $t^{1/3}$ (see (37)). Moreover, taking (38) and (41) into account, it is predicted for large values of t that the standard deviation of the centered and rescaled variable $Z_\pi(t) := t^{-1/3}(F_0(t) - J_\pi t)$ is given by

$$\sqrt{\mathrm{Var}(Z_\pi(t))} = C\pi^{1/3}(1-\pi)^{1/6}, \qquad t \text{ large},$$

with C independent of π. The result of Johansson forcefully reaffirms the KPZ conjecture for dTASEP. Even more is true: in the limit $t \to \infty$, the random variables $Z_\pi(t)/(\pi^{1/3}(1-\pi)^{1/6})$ not only have the same second moments for all $0 < \pi < 1$ but they converge to exactly the same probability distribution.

One may think of this result in analogy to the central limit theorem. There one considers independent, identically distributed random variables X_i. The quantities for which we draw the analogy to the fluxes $F_0(n)$ are the partial sums $S_n = X_1 + \cdots + X_n$. Under some weak assumptions on the distribution of the X_i's one has with probability 1 that S_n/n converges to the expectation $\mu := \mathbb{E}(X_1)$ for $n \to \infty$ (law of large numbers) and that the rescaled random variables $n^{-1/2}(S_n - n\mu)$ tend to a Gaussian distribution (central limit theorem).

In contrast to the central limit theorem, the rescaled variables Z_π do not converge to a Gaussian distribution. Indeed, and quite surprisingly, the limiting distribution is given by the Tracy–Widom distribution of random matrix theory.

Reminder 5.3. (Tracy–Widom distribution)

The Gaussian unitary ensemble GUE is defined as a sequence $\mathbb{P}_N$ of Gaussian probability measures on $N \times N$ Hermitian matrices of the form

$$\mathrm{d}\mathbb{P}_N(M) = \frac{1}{Z_N} \mathrm{e}^{-\mathrm{tr}(M^2)}\, \mathrm{d}M,$$

where Z_N denotes the norming constant and $\mathrm{d}M$ abbreviates the product Lebesgue measure on the real and imaginary parts of the entries of M respecting the Hermitian symmetry, i.e. $\mathrm{d}M = \Pi_i \mathrm{d}M_{ii}\ \Pi_{i<j} \mathrm{d}\,\mathrm{Re}(M_{ij})\, \mathrm{d}\,\mathrm{Im}(M_{ij})$. Denote by $\lambda_1(M)$ the largest eigenvalue of M which is a random object. $\lambda_1(M)$ is expected to be at $\sqrt{2N}$ and the fluctuations are of order $N^{-1/6}$. One can prove that the appropriately rescaled largest eigenvalue converges to some distribution that is now called the Tracy–Widom distribution, i.e. for $s \in \mathbb{R}$ we have

$$\mathbb{P}_N\left(\frac{\lambda_1(M) - \sqrt{2N}}{(8N)^{-1/6}} \leqslant s\right) \longrightarrow TW_2(s) \qquad \text{as } N \to \infty.$$

The function TW_2 can be expressed in terms of the Hastings–McLeod solution of the Painlevé II equation [127] or by Fredholm determinants of integral operators with the Airy kernel (see section 7.2 for more details). Note that the subindex 2 of TW_2 is related to the fact that GUE is a β-random matrix ensemble with $\beta = 2$. Roughly speaking a β-ensemble is an ensemble where the joint distribution of eigenvalues is of the form

$$\mathrm{d}\mathbb{P}_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{\hat{Z}_N} |\Delta(\lambda)|^\beta \prod_{j=1}^{N} w_N(\lambda_j)\, \mathrm{d}\lambda_j$$

and Δ denotes the Vandermonde determinant (cf section 6.4). See [93] for a general reference on random matrix theory. The densities of the Tracy–Widom distributions TW_1 and TW_2 are displayed in figure 3(a) of section 8.

For our model, dTASEP with step initial data, the theorem of Johansson [59] implies

Theorem 5.4. *Let* $0 < \pi < 1$. *Set* $J := \frac{1}{2}(1 - \sqrt{1-\pi})$ *and* $V := 2^{-4/3}\pi^{1/3}(1-\pi)^{1/6}$. *Then, for all* $s \in \mathbb{R}$ *we have*

$$\lim_{t\to\infty} \mathbb{P}_\pi \left(\frac{F_0(t) - Jt}{Vt^{1/3}} \leqslant s \right) = 1 - TW_2(-s). \tag{43}$$

Remark 5.5. As was noted above theorem 5.4 affirms and strengthens the KPZ predictions. In particular, the scaling of the flux is precisely that expected from the KPZ theory. Comparison with (41) shows that $V = (A^2|\lambda|/2)^{1/3}$ in the notation of section 4.

The results of Johansson in [59] are more general than stated above. The mean and fluctuations of the particle flux are described not only at the origin and continuous (in time) TASEP is also considered by letting π tend to 0 and by rescaling time in an appropriate manner.

We would like to emphasize that Johansson's proof of theorem 5.4 does not make any use of the considerations regarding the hydrodynamic limit and the KPZ conjecture as presented above. Instead, the problem treated in theorem 5.4 should be viewed as a very special one within the class of models considered in sections 3 and 4. This problem has the attractive feature that it is exactly solvable by a series of beautiful and non-obvious observations which will be described in the following two sections.

6. Proof of theorem 5.4—part I: combinatorics

We begin our discussion of the proof of theorem 5.4 by relating the flux $F_0(t)$ to another random variable. For $j, k \in \mathbb{N}$ denote

$$T(j,k) := \min\{t \in \mathbb{N} : x_j(t) = k + 1 - j\},$$

that is, the time by which particle j, that starts at site $x_j(0) = -j$, has just completed its $(k+1)$st jump. Observe that at time $T_k := T(k,k)$ we have

$$x_0(T_k) > x_1(T_k) > \cdots > x_k(T_k) = 1 > 0 \geqslant x_{k+1}(T_k) > \cdots.$$

Thus, at time T_k exactly the first $k+1$ particles $0, 1, \ldots, k$ have already jumped from site 0 to site 1 and $F_0(T_k) = k+1$. Moreover, for times $t < T_k$ we have $F_0(t) \leqslant k$. This implies the relation

$$\mathbb{P}_\pi(F_0(t) \leqslant k) = \mathbb{P}_\pi(T_k > t) = 1 - \mathbb{P}_\pi(T(k,k) \leqslant t). \tag{44}$$

In this section we outline how the explicit formula (60) of lemma 6.3 below for the probability distribution of $T(k, k)$ can be derived. By a series of bijections we map our combinatorial model via waiting times and random words to semistandard Young tableaux, a classical object of combinatorics and representation theory where explicit formulas for counting are available. The asymptotic analysis of formula (60) for $\mathbb{P}_\pi(T(k,k) \leqslant t)$ is discussed in section 7.

6.1. From discrete TASEP to waiting times

We introduce an equivalent description of the dynamics of the particle system by a table of waiting times. For $j, l \in \mathbb{N}$ we denote

$$w_{j,l} := \text{ number of times particle } j \text{ decides to stay on site } l - j$$
$$\text{after it becomes possible to jump to site } l - j + 1.$$

For example, in the case of figure 2 we can determine the following entries of the matrix $(w_{j,l})$ of waiting times:

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 1 & \dots \\ 2 & 0 & 0 & 1 & 3 & \dots \\ 1 & 2 & 0 & 2 & ? & \dots \\ 2 & 0 & 2 & 1 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{45}$$

The key observation for computing $T(k, k)$ from the table of waiting times is the following recursion for $T(j, k)$:

$$T(j,k) = 1 + w_{j,k} + \begin{cases} 0, & \text{if} \quad j = k = 0 \\ T(j, k-1), & \text{if} \quad j = 0, k > 0 \\ T(j-1, k), & \text{if} \quad j > 0, k = 0 \\ \max(T(j-1,k), T(j,k-1)), & \text{if} \quad j, k > 0. \end{cases} \tag{46}$$

Indeed, to compute the time it takes the jth particle to complete its $(k+1)$st jump one needs to add $1 + w_{j,k}$ to the time when this jump became possible. For this jump to become possible, particle j has to be on site $k-j$ (happens at $T(j, k-1)$) and particle $j-1$ must have emptied the neighboring site $k-j+1$ (happens at time $T(j-1, k)$). It is obvious from (46) that in order to compute $T(k, k)$ one only needs to know the $(k+1) \times (k+1)$ top-left section of the table of waiting times $(w_{j,l})_{0 \leqslant j,l \leqslant k}$.

Relation (46) allows us to prove the following formula for $T(j, k)$ by induction on $(j+k)$:

$$T(j,k) = j + k + 1 + \max_{\mathcal{P} \in \Pi_{j,k}} \left(\sum_{s \text{ on } \mathcal{P}} w_s \right). \tag{47}$$

Here $\Pi_{j,k}$ denotes the set of paths $\mathcal{P}$ in the table of waiting times that connect the $(0, 0)$-entry with the (j, k)-entry and satisfy the additional condition that only steps to the right-neighbor and to the neighbor downstairs are permitted. More formally, we may write

$$\Pi_{j,k} = \big\{ (s_0, \dots, s_{j+k}) \in (\mathbb{N} \times \mathbb{N})^{j+k+1} : s_0 = (0,0),\, s_{j+k} = (j,k) \text{ and } s_i - s_{i-1} \in \{(1,0),(0,1)\} \text{ for all } 1 \leqslant i \leqslant j+k \big\} .$$

For $\mathcal{P} = (s_0, \dots, s_{j+k}) \in \Pi_{j,k}$ we understand

$$\sum_{s \text{ on } \mathcal{P}} w_s := \sum_{i=0}^{j+k} w_{s_i}.$$

We illustrate formula (47) with our running example. The corresponding table of waiting times displayed in (45) has nine paths in $\Pi_{3,3}$ that maximize the sum of waiting times. Two of them are

$$\mathcal{P}_1 : (0,0) \to (0,1) \to (0,2) \to (0,3) \to (1,3) \to (2,3) \to (3,3) \tag{48}$$

$$\mathcal{P}_2 : (0,0) \to (1,0) \to (2,0) \to (2,1) \to (2,2) \to (3,2) \to (3,3) \tag{49}$$

and we have

$$\sum_{s \text{ on } \mathcal{P}_1} w_s = \sum_{s \text{ on } \mathcal{P}_2} w_s = 8.$$

Formula (47) then yields for the time when the fourth particle $j = 3$ has just completed its fourth jump $T(3, 3) = 3 + 3 + 1 + 8 = 15$ which is easily verified from figure 2.

Remark 6.1. The probabilistic model we have arrived at, i.e. to search for right- and downward paths that maximize the total waiting time, is also known as the last passage percolation problem and that is precisely the model studied in the paper [59] of Johansson. Interpreting $w_{j,l}$ as potential energies this can also be considered as the problem of zero temperature directed polymers in a random medium [55, 64, 75, 76, 80].

We are now ready to compute $P_\pi(T(k,k) \leqslant t)$ in terms of the table of waiting times. Recall from (47) that $T(k,k)$ is completely determined by the top-left $(k+1) \times (k+1)$ corner of the table. Moreover, and again by (47), a table of waiting times corresponds to a particle dynamics with $T(k,k) \leqslant t$ if and only if the top-left corner belongs to the set $W(k,t)$ which we define to be the set of $(k+1) \times (k+1)$ matrices $(w_{j,l})$ with entries that are non-negative integers and which have the additional property

$$\max_{\mathcal{P} \in \Pi_{k,k}} \left(\sum_{s \text{ on } \mathcal{P}} w_s \right) \leqslant t - 2k - 1. \tag{50}$$

It is straightforward to determine the probability that the top-left corner of the table of waiting times agrees with any given element Q of $W(k,t)$. Indeed, we only need to count the total number of decisions to either jump (always equals $(k+1)^2$) or to stay (equals the sum of all entries of Q which we denote by $|Q|_1$) whenever a jump is not prohibited by the exclusion property. In summary we obtain the following formula:

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \sum_{Q \in W(k,t)} \pi^{(k+1)^2} (1-\pi)^{|Q|_1}. \tag{51}$$

6.2. *From waiting times to random words*

We associate with any $(k+1) \times (k+1)$ matrix $Q = (w_{j,l})$ of waiting times the sequence of pairs $(j,l)_{0 \leqslant j,l \leqslant k}$, listed in lexicographical order, where the value of $w_{j,l}$ determines how often the index (j,l) appears in this list. In the case of Q being the top-left 4×4 submatrix in (45) the corresponding sequence of pairs reads

$$\begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 0 & 0 & 3 & 0 & 1 & 1 & 3 & 3 & 0 & 0 & 2 & 2 & 3. \end{array} \tag{52}$$

We may consider this list of pairs as a list of 17 two-letter words from the alphabet $\{0,1,2,3\}$ in lexicographical order. This explains the term 'random words' often used in this context. Observe that any right-downward path $\mathcal{P} \in \Pi_{3,3}$ corresponds to a subsequence in this list of 17 two-letter words where both the first and the second row are weakly increasing. A little more thought shows that the quantity $\max_{\mathcal{P} \in \Pi_{3,3}} \left(\sum_{s \text{ on } \mathcal{P}} w_s \right)$ is given by the length of the longest subsequence that is weakly increasing in both rows. The sequence (52) has nine such subsequences of maximal length 8. The subsequences corresponding to the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ of (48) read

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3' \end{array} \quad \text{and} \quad \begin{array}{cccccccc} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3. \end{array}$$

Formula (51) translates into

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \sum_{\phi \in D(k,t)} \pi^{(k+1)^2} (1-\pi)^{\text{length of } \phi}, \tag{53}$$

where $D(k,t)$ is the set of finite sequences ϕ of lexicographically ordered two-letter words from the alphabet $\{0,1,\ldots,k\}$ and for which the length of the longest subsequence of ϕ that increases weakly in both letters is at most $t - 2k - 1$. By the Robinson–Schensted–Knuth correspondence we may enumerate the set $D(k,t)$ conveniently in terms of semistandard Young tableaux. This is the content of the next section.

6.3. From random words to semi standard young tableaux

The Robinson–Schensted correspondence provides a bijection between permutations and standard Young tableaux that is well known in combinatorics and in the representation theory of the permutation group. We now describe the extension of this algorithm to random words which was introduced by Knuth [67]. The basic algorithm that needs to be understood first is the row insertion process. Suppose we have a weakly increasing sequence of integers, e.g. 0 0 1 1 1 3. We insert an integer r into this row by the following set of rules. If $r \geqslant 3$ we simply append r at the end of the row. In the case $r < 3$ we replace the unique number s in the row that is strictly bigger than r such that after the replacement the sequence is still weakly increasing. We say that we have inserted r by bumping s. For the sequence 0 0 1 1 1 3 insertion of r leads to

r	Sequence after insertion of r	Bumped number
0	0 0 0 1 1 3	1
1	0 0 1 1 1 1	3
2	0 0 1 1 1 2	3
3	0 0 1 1 1 3 3	no number bumped

We now describe the procedure how a lexicographically ordered list of two-letter random words is transformed into a pair of tableaux. In a first step, one only considers the sequence of the second letters of the random words in order to build the first tableau. To this end one inserts (see the row insertion process described above) the second letters of the words, one after the other, into the first row of the tableau that has been created before. In case a number is bumped, the bumped number is inserted into the second row of the tableau. In case bumping occurs again in the second row, we insert the newly bumped number into the third row of the tableau. This process is repeated until the bumping ends. In our example (52), this process leads to the following sequence of tableaux:

$$
\begin{array}{l}
1 \xrightarrow{1} 1\ 1 \xrightarrow{1} 1\ 1\ 1 \xrightarrow{2} 1\ 1\ 1\ 2 \xrightarrow{0} \begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & & & \end{array} \xrightarrow{0} \\[2ex]
\begin{array}{cccccc} 0 & 0 & 1 & 2 & & \\ 1 & 1 & & & & \end{array} \xrightarrow{3} \begin{array}{cccccc} 0 & 0 & 1 & 2 & 3 & \\ 1 & 1 & & & & \end{array} \xrightarrow{0} \begin{array}{cccccc} 0 & 0 & 0 & 2 & 3 & \\ 1 & 1 & 1 & & & \end{array} \xrightarrow{1} \\[2ex]
\begin{array}{cccccc} 0 & 0 & 0 & 1 & 3 & \\ 1 & 1 & 1 & 2 & & \end{array} \xrightarrow{1} \begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 2 & 3 & \end{array} \xrightarrow{3} \begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 2 & 3 & \end{array} \xrightarrow{3} \\[2ex]
\begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 2 & 3 & \end{array} \xrightarrow{0} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 3 & \\ 2 & & & & & \end{array} \xrightarrow{0} \\[2ex]
\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 & \\ 2 & 3 & & & & \end{array} \xrightarrow{2} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 3 & & & & \end{array} \xrightarrow{2} \\[2ex]
\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 3 & & & & \end{array} \xrightarrow{3} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 3 \\ 2 & 3 & & & & \end{array}
\end{array}
$$

Using this procedure we have obtained a semistandard Young tableau with 17 entries.

Definition 6.2. *By a semistandard Young tableau (SSYT) we understand a tableau $\mathcal{T}$ of a finite number of integers that are weakly increasing in each row and strictly increasing in each column. The shape $\lambda = sh(\mathcal{T})$ of $\mathcal{T}$ is denoted by the sequence of row lengths $(\lambda_0, \lambda_1, \ldots)$ that is required to be a weakly decreasing sequence of non-negative integers. Furthermore, we set $|\lambda| := \sum_i \lambda_i$ to be the total number of cells in the tableau.*

In order to see that the procedure described above always leads to a SSYT, one only needs to convince oneself that adding one number to a SSYT will result in a SSYT with one more cell. The key observation here is that in the bumping procedure a number can only move downward or left-downward.

In our running example the final SSYT $\mathcal{T}^*$ has shape $(8, 7, 2, 0, 0, \ldots)$ and it is no coincidence but a theorem that the length of the first row λ_0 equals the length of the longest weakly increasing subsequence that we have seen to be 8. This fact can be proven in general by showing inductively that at every step of the procedure the length of the longest weakly increasing subsequence (in both letters) up to some word in the list is given by the position at which the second letter of the word is inserted in the first row of the tableau. In our example (52) the length of the longest weakly increasing subsequence up to the 15th word $(3, 2)$ is 6 and correspondingly 2 is inserted at the sixth position of the first row.

Note that the sequence (52) is not the only one that leads to the final tableau $\mathcal{T}^*$. For example, the sequence

$$\begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 & 3 & 0 & 0 & 1 & 1 & 3 & 0 & 0 & 0 & 2 & 2 \end{array}$$

leads to the same $\mathcal{T}^*$. However, and this is the central message of the Robinson–Schensted–Knuth correspondence, one may encode the sequence of random words (52) in a unique way if one records in addition how the tableau grows and if one remembers the first letters of the random words that have so far been neglected. This information is all encoded in the second tableau. We now demonstrate how to build this second tableau with our running example. As a first step we record for the above described procedure which cell has been added to the SSYT at which step:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 7 & 11 & 12 & 17 \\ 5 & 6 & 8 & 9 & 10 & 15 & 16 & \\ 13 & 14. & & & & & & \end{array} \tag{54}$$

Since we also need to remember the first letters of our 17 random words it is natural to replace the entries in (54) in the following way. We note that the first four words in (52) have first letter 0 and we therefore replace 1, 2, 3, 4 each by 0. The next three words have first letter 1 and we replace 5, 6, 7 each by 1. Then there are five words starting with letter 2, leading us to replace 8, 9, 10, 11, 12 each by 2. The remaining five entries 13, 14, 15, 16, 17 are each replaced by 3. This leads to the tableau

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & . \\ 3 & 3 & & & & & & \end{array} \tag{55}$$

We have arrived at a SSYT $\mathcal{U}^*$ that clearly has the same shape as $\mathcal{T}^*$. In order to prove that the second tableau always yields a SSYT there is only one non-obvious property to verify, which is the strict increase in each column. To see this recall that for words (a_1, a_2), (b_1, b_2) the number b_2 can only bump a_2 if $a_2 > b_2$ which implies by the lexicographical ordering of the list that $b_1 > a_1$. Therefore, in the second tableau, b_1 will be located below or left-below a_1 and by the weak increase within each row this suffices.

In summary we have mapped the sequence (52) of random words to the pair of SSYTs $(\mathcal{T}^*, \mathcal{U}^*)$

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 3 & 3 & \\ 2 & 3 & & & & & & \end{array}, \qquad \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & \\ 3 & 3 & & & & & & \end{array}$$

of equal shape λ. It is an instructive exercise to reconstruct sequence (52) from the pair of SSYTs. In fact, the proof that the above described procedure maps lexicographically ordered two-letter words bijectively to pairs of SSYTs of equal shape can be given by an explicit description of the inverse map. This bijection has two more features that are of interest to us. First, $|\lambda|$ equals the number of words in our list (=17 in our running example). Second, the length of a longest weakly increasing subsequence is exactly given by the length of the first row λ_0 (= 8 in our example) as we have already observed above (see [67] for details).

This implies that the set $D(k, t)$ (cf (53)) is bijectively mapped onto the set of all pairs $(\mathcal{T}, \mathcal{U})$ of SSYTs of equal shape λ satisfying

$$t - 2k - 1 \geqslant \lambda_0 \geqslant \lambda_1 \geqslant \cdots$$

with entries from $\{0, 1, \ldots, k\}$. Note that we have $\lambda_{k+1} = 0$ because entries in each column are strictly increasing. We therefore arrive at

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \sum_{t-2k-1 \geqslant \lambda_0 \geqslant \ldots \geqslant \lambda_k \geqslant 0} \pi^{(k+1)^2} (1-\pi)^{|\lambda|} L(\lambda, k)^2, \tag{56}$$

where $L(\lambda, k)$ denotes the number of SSYTs of shape $\lambda = (\lambda_0, \ldots, \lambda_k, 0, \ldots)$ and with entries from $\{0, 1, \ldots, k\}$. We have now derived a representation for $\mathbb{P}_\pi(T(k,k) \leqslant t)$ involving the combinatorial quantity $L(\lambda, k)$ that can be computed explicitly.

6.4. Schur polynomials and an explicit formula for the distribution of T(k, k)

There is a beautiful argument using Schur polynomials s_λ that provides a formula for $L(\lambda, k)$. This argument is explained in the appendix of [110] and we do not repeat it here (see in addition [108, corollary 4.6.2] for a derivation of the representation [110, (A.5)] of s_λ by determinants). The result is

$$L(\lambda, k) = \prod_{0 \leqslant i < j \leqslant k} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{57}$$

Introducing the new variables $y_i := \lambda_i - i + k$ and denoting the Vandermonde determinant by $\Delta(y) = \prod_{0 \leqslant i < j \leqslant k} (y_j - y_i)$ we obtain

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = C_{\pi,k} \sum_{t-k-1 \geqslant y_0 > \cdots > y_k \geqslant 0} \Delta(y)^2 \prod_{i=0}^{k} (1-\pi)^{y_i}, \quad \text{where} \tag{58}$$

$$C_{\pi,k} := \pi^{(k+1)^2} (1-\pi)^{-k(k+1)/2} \prod_{0 \leqslant i < j \leqslant k} \frac{1}{(j-i)^2}. \tag{59}$$

Observe that each term in the sum is a symmetric function in y that vanishes if two components agree. This leads to the final formula in this section for the probability distribution of $T(k, k)$ which is closely related to the distribution of the flux $F_0(t)$ via (44).

Lemma 6.3.

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \frac{C_{\pi,k}}{(k+1)!} \sum_{\substack{y \in \mathbb{Z}^{k+1} \\ 0 \leqslant y_i \leqslant t-k-1}} \Delta(y)^2 \prod_{i=0}^{k} (1-\pi)^{y_i}. \tag{60}$$

This formula should be compared with the formula for the distribution of the largest eigenvalue of the Gaussian unitary ensemble (cf reminder (5.3))

$$\mathbb{P}_N(\lambda_1(M) \leqslant \Lambda) = \frac{1}{\hat{Z}_N} \int_{(-\infty,\Lambda]^N} \Delta(y)^2 \prod_{j=1}^{N} e^{-y_j^2} \, \mathrm{d}y \tag{61}$$

with some appropriate norming constant $\hat{Z}_N$. Observe that this formula has exactly the same structure as (60). The role played by the measure $\mathrm{e}^{-x^2}\,\mathrm{d}x$ for GUE is taken by the discrete measure $\sum_{j=0}^{\infty}(1-\pi)^j \delta_j$ supported on $\mathbb{N}$ for dTASEP. Here δ_j denotes the δ-distribution concentrated at j. In the next section we discuss how the method of orthogonal polynomials can be used to analyze the asymptotics of such types of high-dimensional integrals.

7. Proof of theorem 5.4—part II: asymptotic analysis

In the previous section we have derived formulae (60) and (44) for the probability distribution of the flux $F_0(t)$. We now need to analyze this formula asymptotically in a regime where t and k both become large and $k \approx Jt + Vt^{1/3}s$ with s being an arbitrary but fixed real number and J, V being defined as in theorem 5.4. The key to the analysis is the observation that the right-hand side of (60) is structurally the same as the standard formula for the probability distribution of the largest eigenvalue of GUE (61) and the method of orthogonal polynomials (see section 7.1) can be applied. The role played by Hermite polynomials for GUE will be taken by Meixner polynomials in our model. In both cases it is the behavior of the orthogonal polynomials of large degree in a vicinity of their respective largest zero that matters in the asymptotic analysis. After appropriate rescaling this behavior can be described in terms of Airy functions for both Hermite and Meixner polynomials (see section 7.2). On a technical level this explains the occurrence of the Tracy–Widom distribution TW_2 for GUE as well as for dTASEP with step initial conditions. We include in section 7.3 a brief discussion of the universal behavior of orthogonal polynomials.

7.1. The method of orthogonal polynomials following an approach of Tracy and Widom

Almost 50 years ago, Gaudin and Mehta had introduced orthogonal polynomials to random matrix theory in order to study local eigenvalue statistics. A very transparent version of the now so-called *method of orthogonal polynomials* is due to Tracy and Widom [128] and we will briefly outline their approach for our situation. See also the recent book of Deift and Gioev [35, chapter 4] for a self-contained presentation of the method of orthogonal polynomials with some remarks on the history of the method. For us the method allows us to express the right-hand side of (60) in terms of a Fredholm determinant of an operator where the (discrete) integral kernel is given in terms of Meixner orthogonal polynomials.

We begin our discussion by introducing the discrete weight

$$w_\pi(x) := \begin{cases} 0, & \text{if} \quad x < 0 \\ (1-\pi)^x, & \text{if} \quad x \geqslant 0 \end{cases}, \qquad x \in \mathbb{Z}.$$

For a moment we let $(q_l)_{l\geqslant 0}$ be any sequence of polynomials with q_l being of degree l with (nonzero) leading coefficient γ_l. Setting $\varphi_l(x) := q_l(x)\sqrt{w_\pi(x)}$ and using the definition of the Vandermonde determinant together with some basic properties of determinants we have for $y \in \mathbb{N}^{k+1}$

$$\left[\det(\varphi_l(y_i))_{0\leqslant i,l\leqslant k}\right]^2 = (\gamma_0 \dots \gamma_k)^2 \Delta(y)^2 \prod_{i=0}^{k} (1-\pi)^{y_i}.$$

Furthermore, we set $I_s := [s, \infty)$ and denote by $\mathbf{1}_{I_s}$ its characteristic function that takes the value 1 on I_s and 0 on $\mathbb{R} \setminus I_s$. We may then rewrite (60) in the form

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \frac{C_{\pi,k}}{(\gamma_0 \dots \gamma_k)^2 (k+1)!} \sum_{y\in\mathbb{Z}^{k+1}} [\det(\varphi_l(y_i))]^2 \prod_{i=0}^{k} (1 - \mathbf{1}_{I_{t-k}}(y_i)).$$

An identity due to Andréief (see e.g. [35, (3.3)]) adapted to our context reads

$$\sum_{y\in\mathbb{Z}^{k+1}} [\det(\varphi_j(y_i))][\det(\varphi_l(y_i))] \prod_{i=0}^{k} f(y_i) = (k+1)! \det\left(\sum_{x\in\mathbb{Z}} \varphi_j(x)\varphi_l(x) f(x)\right).$$

One may prove this formula using the Leibniz sum for determinants. This allows us to write the distribution of $T(k, k)$ as a determinant

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \frac{C_{\pi,k}}{(\gamma_0 \dots \gamma_k)^2} \det S,$$

where S denotes the $(k+1) \times (k+1)$ matrix with entries

$$S_{j,l} = \sum_{x\in\mathbb{Z}} \varphi_j(x)\varphi_l(x) \left(1 - \mathbf{1}_{I_{t-k}}(x)\right), \qquad 0 \leqslant j, \quad l \leqslant k.$$

So far the choice of the polynomials q_l of degree l was arbitrary. Now we choose $(q_l)_l$ to be the sequence of normalized orthogonal polynomials with respect to the discrete measure $\sum_{x\in\mathbb{Z}} w_\pi(x)\delta_x$ which belongs to the class of Meixner polynomials. We have

$$\sum_{x\in\mathbb{Z}} \varphi_j(x)\varphi_l(x) = \sum_{x\in\mathbb{Z}} q_j(x)q_l(x)w(x) = \delta_{j,l}$$

for $j, l \in \mathbb{N}$. Hence $S = I - R(t-k)$ with

$$R(s)_{j,l} = \sum_{x\in\mathbb{Z}} \varphi_j(x)\varphi_l(x)\mathbf{1}_{I_s}(x) = \sum_{x\geqslant s} \varphi_j(x)\varphi_l(x).$$

In summary, we have so far derived

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \frac{C_{\pi,k}}{(\gamma_0 \dots \gamma_k)^2} \det(I - R(t-k)). \tag{62}$$

The prefactor $C_{\pi,k}(\gamma_0 \dots \gamma_k)^{-2}$ can be seen to equal 1 by considering equation (62) for fixed k in the limit $t \to \infty$. Thus,

$$\mathbb{P}_\pi(T(k,k) \leqslant t) = \det(I - R(t-k)). \tag{63}$$

The final idea in the argument of Tracy–Widom is to write $R(s)$—considered as a linear map $\mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$—as a product $R(s) = A(s)B(s)$, with

$$B(s) : \mathbb{R}^{k+1} \to \ell_2(\mathbb{Z} \cap I_s), \qquad (u_j)_{0\leqslant j\leqslant k} \mapsto \sum_{j=0}^{k} u_j \varphi_j|_{I_s}$$

$$A(s) : \ell_2(\mathbb{Z} \cap I_s) \to \mathbb{R}^{k+1}, \qquad f \mapsto \left(\sum_{x\geqslant s} f(x)\varphi_l(x)\right)_{0\leqslant l\leqslant k}.$$

Applying the formula $\det(I - AB) = \det(I - BA)$ that holds in great generality (see e.g. [35, (3.1)]) we have derived the following Fredholm determinant formula for $\mathbb{P}_\pi(T(k,k) \leqslant t)$.

Lemma 7.1. $\mathbb{P}_\pi(T(k,k) \leqslant t) = \det(I - \Sigma_k(t-k))$, *where*

$$\Sigma_k(s) : \ell_2(\mathbb{Z} \cap I_s) \to \ell_2(\mathbb{Z} \cap I_s), \quad f \mapsto \left(\sum_{y \geqslant s} \sigma_k(x,y) f(y) \right)_{x \geqslant s}$$

and σ_k *denotes the reproducing kernel* $\sigma_k(x,y) := \sum_{j=0}^{k} \varphi_j(x)\varphi_j(y)$ *with respect to the Meixner polynomials.*

It may seem somewhat strange to convert (62) that involves a determinant of some finite-size matrix $I-R$ into a formula that involves the computation of a Fredholm determinant of an operator acting on the infinite-dimensional space $\ell_2(\mathbb{Z} \cap I_s)$. However, one has to keep in mind that we are interested in an asymptotic result with $k \to \infty$. Hence, the size of $I-R$ goes to infinity and it is not at all clear how to perform the asymptotic analysis of the determinants. In contrast, the operator $I - \Sigma_k$ acts on the same space $\ell_2(\mathbb{Z} \cap I_s)$ for all k and it is only the reproducing kernels σ_k that depended on k. As will be discussed below, the kernels σ_k are amenable to asymptotic analysis. In fact, due to the Christoffel–Darboux formula for orthogonal polynomials we may express σ_k just in terms of φ_k and φ_{k+1}. For large values of k the behavior of these functions is rather well understood. For example, if x is somewhat larger than the largest zero of φ_k, then $|\varphi_k(x)|$ is very close to zero. This implies that for values of $t-k$ that are somewhat larger than the largest zeros of φ_k and φ_{k+1}, the operator $\Sigma_k(t-k)$ is negligible and thus $\mathbb{P}_\pi(T(k,k) \leqslant t)$ is very close to 1. If one reduces the value of $t-k$ to lie in a vicinity of the largest zero of φ_k (which is also close to the largest zero of φ_{k+1}) then the functions φ_k and φ_{k+1}, appropriately rescaled, are described to leading order by Airy functions. In section 7.2 we will use the just-mentioned properties of Meixner polynomials to complete the proof of theorem 5.4.

As was noted in the last paragraph of section 6.4, the formula for the distribution of the largest eigenvalue of GUE (61) is structurally the same as formula (60) for the distribution of $T(k,k)$ and the arguments described in this section can be applied in an analogous way. The only difference is that we need to use Hermite polynomials instead of Meixner polynomials and that the summation operator Σ_k is to be replaced by an integral operator with a kernel that is given by the reproducing kernel for Hermite polynomials up to degree $N-1$ (N as in $\mathbb{P}_N$, cf reminder (5.3)). As in the Meixner case, the leading-order behavior of Hermite polynomials near their largest zero is described by Airy functions. On a technical level this is the reason why the fluctuation of the flux in dTASEP with step initial conditions follows asymptotically the same distribution as the fluctuation of the largest eigenvalue of GUE. It is no coincidence that Meixner polynomials and Hermite polynomials of large degree look locally the same when rescaled appropriately. In fact, large classes of orthogonal polynomials display the same local behavior. We will comment on this universality property of orthogonal polynomials in section 7.3.

7.2. Completing the proof of theorem 5.4

We have argued above that in order to complete the proof of theorem 5.4 we need to determine the asymptotic behavior of the reproducing kernel σ_k (see lemma 7.1) in a vicinity of the largest zero of φ_k. This requires some detailed analysis that we are not going to present here and we refer the reader to [59, section 5]. See also section 7.3 for a few general remarks on

the asymptotic analysis of orthogonal polynomials. We start with some notation. The Airy function can be defined for $x \in \mathbb{R}$ by

$$\mathrm{Ai}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp(\mathrm{i}[x(t + \mathrm{i}s) + (t + \mathrm{i}s)^3/3])\, \mathrm{d}t \tag{64}$$

with an arbitrary choice of $s > 0$. The Airy kernel is given by

$$A(x, y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}, \qquad x, y \in \mathbb{R},$$

with the obvious interpretation on the diagonal $x = y$. The Tracy–Widom distribution for $\beta = 2$ can then be expressed as a Fredholm determinant

$$\mathrm{TW}_2(s) := \det(I - A)|_{L^2[s,\infty)}, \qquad s \in \mathbb{R},$$

where A denotes the integral operator associated with the Airy kernel. Note that one may derive a differential equation for TW_2 leading to another representation [127]

$$\mathrm{TW}_2(s) = \exp\left(-\int_s^\infty (x - s)u(x)^2 \mathrm{d}x\right),$$

where u denotes the Hastings–McLeod solution of the Painlevé II equation $u'' = 2u^3 + xu$ that is singled out from all solutions of this ordinary differential equation by the asymptotic condition $u(x) \sim -\mathrm{Ai}(x)$ for $x \to \infty$. Observe that the Airy function solves the linearized Painlevé II equation $u'' = xu$ with asymptotics $\mathrm{Ai}(x) \sim \frac{\exp(-(2/3)x^{3/2})}{2\sqrt{\pi}x^{1/4}}$ as $x \to \infty$. In an interesting paper [21] Bornemann demonstrates that it is advantageous for numerical evaluations of TW_2 to start from the Fredholm determinant formula rather than using the Hastings–McLeod function.

The result in [59, lemma 3.2] on the reproducing kernel σ_k for Meixner polynomials reads

$$ck^{1/3}\sigma_k(bk + ck^{1/3}\xi, bk + ck^{1/3}\eta) \to A(\xi, \eta) \qquad \text{for} \quad k \to \infty, \tag{65}$$

where $b = \pi^{-1}(1 + \sqrt{1 - \pi})^2$ and $c = \pi^{-1}(1 - \pi)^{1/6}(1 + \sqrt{1 - \pi})^{4/3}$. We can now derive (43) formally. From (44), lemma 7.1, and (65) we learn that we should have

$$t - k = bk + ck^{1/3}(-s)(1 + o(1)) \tag{66}$$

for $t \to \infty$ and with $k = Jt + Vst^{1/3}$. A straightforward calculation shows that this can only be achieved if

$$(b + 1)J = 1 \qquad \text{and} \qquad V(b + 1) = cJ^{1/3}$$

leading to the formulae for J and V as presented in theorem 5.4. Moreover, it is apparent that the remainder $o(1)$ in (66) is of order $\mathcal{O}(t^{-2/3})$. Clearly, the argument just made does not fully prove theorem 5.4 since additional estimates are needed to deduce the convergence of Fredholm determinants from the convergence of the kernels. For details see [59, section 3].

Finally we observe that the linear growth of the mean and the $t^{1/3}$-scaling of the fluctuations of the flux $F_0(t)$ follow directly from condition (66), i.e. from (44), lemma 7.1, and (65) without any reference to the KPZ theory.

7.3. Remarks on the universal behavior of orthogonal polynomials

We have discussed two examples for fluctuations that can be described by the Tracy–Widom distribution, the largest eigenvalue of GUE and the particle flux at the origin of dTASEP with step initial conditions. Both systems can be analyzed by the method of orthogonal

polynomials. The fact that the limiting distribution for the fluctuations agree in both cases is then a consequence of the fact that the corresponding sets of orthogonal polynomials, Hermite and Meixner, have the same asymptotic behavior in a vicinity of their respective largest zeros after appropriate rescaling. This is no coincidence. During the past 10 years many detailed results on various types of orthogonal polynomials have become available that show universal behavior of orthogonal polynomials of large degree on a local scale. In this section we give a rough description of this universal behavior and outline a few approaches how such results can be proved.

Let us assume that the support of the measure of orthogonality α is contained in $\mathbb{R}$. Then, the normalized orthogonal polynomial q_n of degree n with a positive leading coefficient, that is defined through

$$\int_{\mathbb{R}} q_n(x)q_m(x)\,\mathrm{d}\alpha(x) = \delta_{n,m},$$

has n simple real roots which we denote by $x_i^{(n)}$. For many measures of orthogonality α, and in particular in the case of the so-called varying weights which we do not discuss here any further, there exists some natural scaling $x \to \hat{x}$ such that the counting measures $\frac{1}{n}\sum_{i=1}^{n} \delta_{\hat{x}_i^{(n)}}$ associated with the rescaled zeros $\hat{x}_i^{(n)}$ converge for $n \to \infty$ to some measure μ of total mass 1. The support S of μ is compact and is always contained in the support of α. For example, in the case of Hermite polynomials $\mathrm{d}\alpha = \mathrm{e}^{-x^2}\,\mathrm{d}x$, the scaling is given by $\hat{x} = x/n^{1/2}$ and the limiting measure of zeros is given by $\mathrm{d}\mu = \pi^{-1}\sqrt{2-x^2}\,\mathbf{1}_S(x)\,\mathrm{d}x$ with $S = [-\sqrt{2}, \sqrt{2}]$.

One should note that the scaling $x \to \hat{x}$ as well as the measure of zeros μ do depend on the measure of orthogonality α. In order to explain what is universal about orthogonal polynomials we restrict ourselves to the common case that S, the support of μ, consists of a single interval or a finite union of disjoint intervals. It is convenient to describe the behavior of q_n by considering the corresponding functions φ_n that are orthonormal with respect to the Lebesgue measure. For example, in the case $\mathrm{d}\alpha = w(x)\,\mathrm{d}x$ we have $\varphi_n = q_n\sqrt{w}$. In the situation described above the large n behavior of $\varphi_n(x)$ is generically as follows.

For $\hat{x}$ outside S. $\varphi_n(x)$ decays at an exponential rate to zero as $n \to \infty$.

For $\hat{x}$ in the interior of S. $\varphi_n(x)$ is oscillating rapidly (in $\hat{x}$) and can be described to leading order by the cosinus function with slowly varying amplitude and frequency, which only depend on μ.

For $\hat{x}$ close to a boundary point b of S. To leading order $\varphi_n(x)$ can be expressed in terms of special functions. One distinguishes between soft edges (b lies in the interior of the support of α) and hard edges (b is also a boundary point of the support of α). In the first case the density of μ usually vanishes like a square root at b and then the leading order of $\varphi_n(x)$ is described by the Airy function. This is in particular the case for Hermite and Meixner polynomials. In the case of a hard edge the situation is a bit more complicated. In many cases Bessel functions can be used for asymptotic formulas for $\varphi_n(x)$ (see e.g. [81, 82]).

We conclude this section by mentioning a few methods how such asymptotics for φ_n can be proved. We will in particular remark on the appearance of the Airy function.

(I) *Differential equations of second order*

We again discuss Hermite polynomials $\mathrm{d}\alpha = \mathrm{e}^{-x^2}\mathrm{d}x$ as a typical example. The corresponding functions φ_n satisfy the second-order differential equations

$$\varphi_n''(x) + (2n + 1 - x^2)\varphi_n(x) = 0.$$

The WKB analysis of these differential equations shows that the oscillatory region $|x| < \sqrt{2n+1}$ is connected with the exponentially decaying region $|x| > \sqrt{2n+1}$

by Airy functions. This approach can be applied to a number of classical orthogonal polynomials that are known to solve linear differential equations of second order with nice coefficients.

(II) *Representation by contour integrals*

Such representations are known for a number of classical orthogonal polynomials (e.g. for Meixner polynomials, see e.g. [59, section 5]) and can be analyzed using the method of steepest descent. The appearance of the Airy function can be seen from its integral representation (64) which generically provides a normal form at critical points of higher degeneracy.

(III) *Riemann–Hilbert problems*

The characterization of orthogonal polynomials as unique solutions of certain matrix Riemann–Hilbert problems (see [33] and references therein) works in principle for all types of weights and opens in particular the way to analyze non-classical orthogonal polynomials. Here the limiting measure of the zeros μ, which can also be defined as the unique minimizer of a variational problem, yields the key to the asymptotic analysis. In the neighborhoods of boundary points of the support of μ at which the density vanishes like a square root—this is the generic case for a soft edge—Airy functions arise naturally. This method for the asymptotic analysis of orthogonal polynomials was first carried out in [36]. The method works best in the class of analytic weights, but progress has recently been made for weights that have only a finite number of derivatives [91]. Orthogonal polynomials with respect to discrete measures have been analyzed by Riemann–Hilbert techniques in [8].

(IV) *Reproducing kernels*

In recent years universality results for orthogonal polynomials, in particular results on the reproducing kernel (cf lemma 7.1), have been substantially generalized (see [87] and references therein). A very nice view on universality has been introduced in [86] where classical results on reproducing kernels for entire functions of exponential type are being used (see [84] for the Airy kernel).

8. KPZ-universality revisited

In view of the universality conjecture formulated in section 4, one expects that the results derived in the preceding sections for a very special case—the dTASEP with step initial conditions—should carry over, in a quantitative sense, to a much broader class of models. The first explicit demonstration of this idea was presented by Prähofer and Spohn in a series of papers [99–101], where an alternative and independent route linking Ulam's problem to growth models was established[10].

The starting point is the one-dimensional polynuclear growth model (PNG), an interacting particle system on the real line, in which particles (antiparticles) move deterministically at unit speed to the right (left), annihilate upon colliding and are created in pairs according to a two-dimensional Poisson process in space and time [78]. Via the random set of particle creation events the model can be mapped onto the problem of the longest increasing subsequence of a random permutation, which in turn provides a link to the Tracy–Widom distribution [6]. For the case of a droplet growing from a seed (case III of section 4), Prähofer and Spohn show that the resulting fluctuation distribution is identical (under the rescaling prescribed by the KPZ theory) to that obtained by Johansson for the dTASEP.

[10] Another link between the two classes of problems was found by Majumdar and Nechaev [88, 89].

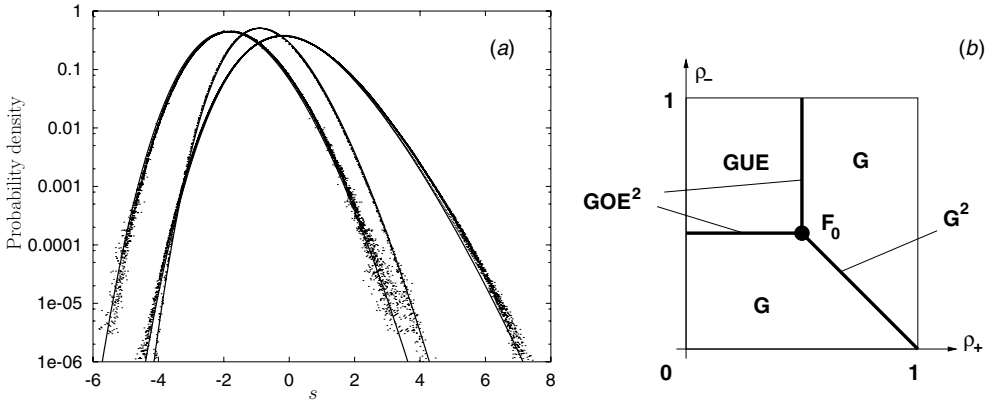

Figure 3. (*a*) The densities of the three universal distribution functions TW_2, TW_1 and F_0 (from left to right). Discrete points show simulation results for the PNG model. Reprinted with permission from [100]. Copyright 2000 by the American Physical Society. (*b*) Phase diagram for the distribution of current fluctuations in the TASEP with Bernoulli step initial conditions. Here G denotes the Gaussian distribution and G^2 (GOE^2) is the distribution of the maximum of two independent Gaussian (TW_1) random variables. Initial particle densities are $\rho_L = \rho_-$ to the left and $\rho_R = \rho_+$ to the right of the origin. Reprinted from [101] with kind permission of Springer Science and Business Media.

Moreover, by imposing suitable boundary conditions [9] and symmetry relations [10] on the set of Poisson points, the cases of flat and rough initial conditions (cases I and II of section 4) can also be handled [100]. For the flat initial condition (case I) the fluctuations are governed by the GOE distribution TW_1, while for the rough initial condition (case II) a new distribution F_0 emerges which so far does not have an interpretation in terms of random matrix theory [9]. The three distributions are depicted in figure 3(*a*).

Since the fundamental works of Johansson, Prähofer and Spohn, the field has developed rapidly, and it is impossible to do justice to the new results in the framework of these lecture notes. In the following subsections we therefore restrict ourselves to briefly outlining the most important directions of research, providing the interested reader with a few key references along which recent advances can be traced.

8.1. The Prähofer–Spohn conjecture and the ASEP

Based on the universality hypothesis, Prähofer and Spohn translated the results obtained for the PNG model into a conjecture for the fluctuations of the particle current through the origin for the TASEP with a general step initial condition (20), where particles are placed to the left (right) of the origin according to a Bernoulli measure with density ρ_L (ρ_R) [101]. The fluctuation phase diagram in the plane of the boundary densities is shown in figure 3(*b*). The overall features of the diagram can be understood from hydrodynamics. First, the Johansson result obtained at $\rho_L = 1$, $\rho_R = 0$ is seen to extend throughout the region $\rho_L > 1/2 > \rho_R$. As explained in section 4, this reflects the fact that the density profile near the origin is independent of the boundary densities in this case. For $\rho_L < 1/2$ and $\rho_R > 1/2$ the application of the hydrodynamic formulae (21) (for $\rho_L > \rho_R$) and (19) (for $\rho_L < \rho_R$) show that the density at the origin becomes ρ_L and ρ_R, respectively. In these cases the intrinsic current fluctuations are masked by the initial fluctuations drifting across the origin, leading to simple Gaussian statistics (regions marked G in the diagram). The line $\rho_R + \rho_L = 1$, $\rho_L < \rho_R$, is special,

because there the shock speed (19) vanishes and the density at the origin shifts randomly between ρ_L and ρ_R. As a consequence, the current is distributed as the maximum of two independent Gaussian random variables (denoted by G^2 in the figure). Similarly, along the lines $\rho_L = 1/2$, $\rho_R < 1/2$, and $\rho_R = 1/2$, $\rho_L > 1/2$, the distribution is that of the maximum of two independent variables drawn from TW_1. Finally, at the point $\rho_L = \rho_R = 1/2$ we have case II behavior governed by the distribution F_0.

A proof of the Prähofer–Spohn conjecture for the TASEP was recently presented by Ben Arous and Corwin [4] (see also [5, 48, 96] for earlier partial results). Moreover, in a remarkable series of papers Tracy and Widom have been able to generalize these results to the (partially) asymmetric exclusion process [130–136]. The generalization is highly nontrivial, because the ASEP for general q is not a determinantal process [133], and it requires a novel set of techniques based on the Bethe ansatz [53, 105, 118].

8.2. *Spatiotemporal scaling*

We have seen in section 4 that the essence of the KPZ conjecture is the universality of height fluctuations when viewed on the appropriate scales defined by the height rescaling (37) and the correlation length (39). In other words, once the average growth shape has been subtracted, one expects that the rescaled fluctuations

$$\bar{h}_t(y) \equiv (A^2|\lambda|t)^{-1/3} h(y(A\lambda^2 t^2)^{1/3}, t)$$

converge for $t \to \infty$ to a universal stochastic process $\mathcal{A}(y)$, whose single-point distribution is one of the random matrix distributions discussed above. The process $\mathcal{A}(y)$ was first explicitly characterized by Prähofer and Spohn for the PNG model in the droplet geometry (case III of section 4), who named it the *Airy process* [102]. Subsequently Sasamoto identified the analogous process for the case of flat initial conditions (case I) [109]. In line with the nomenclature used to designate the corresponding single-point distributions (TW_2 for case III and TW_1 for case I), the two processes are now called $Airy_2$ and $Airy_1$ processes, respectively [23–25, 50]. Whereas the $Airy_2$ process has a natural interpretation in the random matrix context as the motion of the largest eigenvalue in GUE matrix diffusion (Dyson's Brownian motion), the corresponding relation does not hold for the $Airy_1$ process [22]. The corresponding process for initial conditions with stationary roughness (case II) was studied in [7] (see also [58]).

Recently, Ferrari and collaborators have extended the analysis to include correlations between height fluctuations at different times t and $t' > t$. As for the scaling of the height fluctuations themselves, the characteristics of the hydrodynamic equation play a special role for the decay of correlations. Whereas the decay along generic spacetime directions is governed by the correlation length (39), which is of order $t^{2/3}$, along the characteristics the decorrelation time at time t is set by t itself, which implies a much slower decay [31, 32, 49]. This is in accordance with the KPZ phenomenology, which predicts that such correlations should decay as $(t/t')^{\bar{\lambda}}$ with a universal autocorrelation exponent $\bar{\lambda}$ [61, 69]. In contrast to the scaling exponents of single-point height fluctuations introduced in section 4, the autocorrelation exponent depends explicitly on the growth geometry. For a flat initial condition (case I) $\bar{\lambda} = 1$, whereas for a curved cluster (case III) $\bar{\lambda} = 1/3$ [121].

8.3. *KPZ scaling at large*

The results described so far in this section were based on a small set of exactly solvable models, the (T)ASEPs and the PNG model. On the other hand, KPZ universality is expected to hold for a much broader class of interacting particle systems and growth models which

is limited only by the requirement of local, stochastic transition rules and a nonlinear dependence of the particle current (or growth rate) on the particle density (or surface slope) (see section 4). It is therefore gratifying that the class of models for which KPZ universality has been rigorously established—mostly in the sense of finding the exact scaling exponents governing the order of fluctuations—has been greatly expanded in recent years. This has required the development of new techniques that are purely probabilistic in nature and do not rely on the specific analytic structure of the exactly solvable models. The first result of this type was obtained in [28] for the Hammersley process, an exclusion-type process in continuous space which is closely related to the Ulam problem. Similar methods were subsequently applied to a variety of interacting particle systems [15, 16, 103, 104], including a class of zero-range processes[11] with general jump rates that have a non-decreasing, concave dependence on the number of particles [12]. This constitutes a major step on the way to proving KPZ universality in the broadest sense.

8.4. The universality class of the KPZ equation

Ironically, although tremendous advances in the analysis of different representatives of the KPZ universality class were achieved over the past decade, the one-dimensional KPZ equation (33) itself remained rather poorly understood. We noted already that the KPZ equation is mathematically ill-posed because of the highly singular white noise term, and some regularization is needed to make it amenable to rigorous analysis. This can be done by spatial discretization or by constructing the equation through a scaling limit from an asymmetric exclusion process with weak asymmetry [17, 112]. Both approaches have recently been used to prove the correct order of fluctuations in the KPZ equation [13, 111] as well as refined universality in the sense of the Tracy–Widom distribution [3, 113, 114]. Thus, it has finally been established, as it were, that the KPZ equation belongs to its own universality class.

In another line of work an independent, non-rigorous approach to establishing Tracy–Widom universality has been developed which is based on applying the replica method to the path weight $Z(x, t)$ in the Hopf–Cole transformed equation (36). In this approach one computes moments $\mathbb{E}(Z^n)$ with respect to the stochastic force $\zeta(x, t)$, which results in a problem of n bosonic, quantum-mechanical particles interacting through an attractive δ-function potential [55, 62]. While the ground-state energy and wavefunction of this quantum system have been known for a long time, the recent works [27, 42] have succeeded in summing over the full spectrum of excited states of the many-body Hamiltonian.

8.5. Experiments

Despite the wide applicability of the KPZ theory to a large class of stochastic growth models, experimental signatures of KPZ scaling in the real physical world have proven to be surprisingly elusive. Early efforts focused on the investigation of growth-induced (two-dimensional) surface roughness of crystals and thin solid films [70]. However, a detailed consideration of the physical processes governing such growth experiments has revealed that they typically operate in regimes where KPZ asymptotics is practically out of reach [72, 94]. To date, the most thoroughly studied experimental system that displays KPZ scaling is the slow, flameless combustion of paper. By imaging and analyzing the one-dimensional smoldering front, the $t^{1/3}$-scaling of fluctuations [90] as well as the non-Gaussian shape of the corresponding probability distribution [95] were demonstrated.

[11] Zero-range processes were introduced by Spitzer [122] and have been extensively studied in the physics literature, see [44].

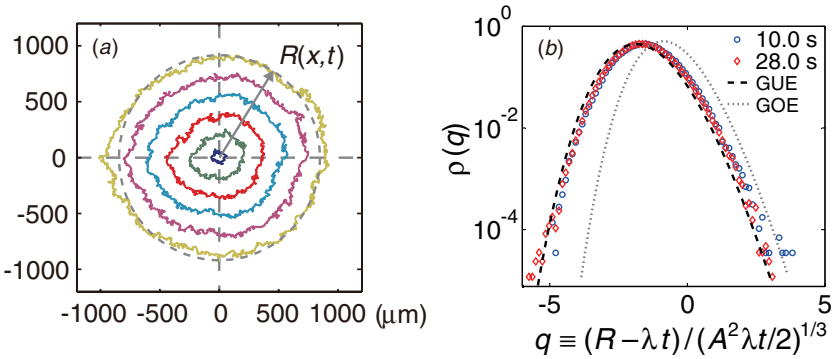

Figure 4. Experimental demonstration of KPZ universality in a thin film of turbulent liquid crystal. (*a*) Outlines of a growing turbulent droplet at time intervals of 5 s. (*b*) Measured distribution of shape fluctuations in comparison to the GUE prediction. Reprinted with permission from [126]. Copyright 2010 by the American Physical Society.

Very recently, a new experimental system involving different phases of driven, turbulent liquid crystal films became available, in which the refined universality predictions of the theory can be tested with unprecedented precision [126]. In figure 4 we show a series of snapshots of the growing turbulent droplet along with the experimentally determined distribution of shape fluctuations. Under the rescaling prescribed by the KPZ theory, the predicted universal GUE distribution is seen to emerge from the data without any adjustable parameters.

9. Integrability and universality

The central model discussed in our paper is the totally asymmetric simple exclusion process (TASEP). This model has two features that motivate our choice. Firstly, it belongs to the class of stochastic interacting particle systems that are useful in the study of transport phenomena in nonequilibrium systems. We are particularly interested in the description of fluctuations around the mean behavior which is governed by a deterministic evolution equation. In general, it is difficult to obtain such detailed information, mainly because the interactions between the particles destroy stochastic independence and the classical central limit theorem cannot be applied. However, and this is the second feature, TASEP is an exactly solvable model and the fluctuations can be analyzed by a series of beautiful and non-obvious observations. We have seen by explicit calculation that the fluctuations of the flux are described by a formula that is structurally the same as the formula for the fluctuations of the largest eigenvalue of matrices from the Gaussian unitary ensemble (GUE). This provides one explanation for the much celebrated link to random matrix theory. There are also links to the theory of integrable systems (see e.g. [33, 139, 140]) that we have not explained at all and that we mention here only in order to give an indication why TASEP is sometimes called an integrable model, even though there is no differential equation to integrate in sight.

A common feature of integrable systems is that their delicate mathematical structure immediately breaks down if the model is changed ever so slightly. One may think that this limits the interest in the corresponding results. However, the recurring experience with integrable systems has been that even though the method of proof is not applicable if the model is changed, the results may persist. In our context a nice example of this principle of universality is the recent work of Tracy and Widom where it is shown by very different

methods that a number of results for TASEP also hold for ASEP. Universality results are also available for large classes of matrix ensembles (see [35, 43] and references therein). Despite these results, the question of universality is a subtle one as can be seen for example from the correlations of the fluctuations (cf [22] and section 8.2). It remains a great challenge to understand the realm of validity of the various laws that have been established for the specific integrable models.

Acknowledgments

This work was supported and inspired by SFB/TR 12 *Symmetries and Universality in Mesoscopic Systems*. We are grateful to H Spohn, P L Ferrari and to an unknown referee for valuable remarks, and to M Prähofer and K A Takeuchi for providing figures.

References

[1] Aldous D and Diaconis P 1999 Longest increasing subsequences: from patience sorting to the Baik–Deift–Johansson theorem *Bull. Am. Math. Soc.* **36** 413–32
[2] Amar J G and Family F 1992 Universality in surface growth—scaling functions and amplitude ratios *Phys. Rev.* A **45** 5378–93
[3] Amir G, Corwin I and Quastel J 2010 Probability distribution of the free energy of the continuum directed random polymer in 1+1 dimensions *Commun. Pure Appl. Math.* (arXiv:1003.0443v2) at press
[4] Ben Arous G and Corwin I 2009 Current fluctuations for TASEP: a proof of the Prähofer–Spohn conjecture *Ann. Probab.* (arXiv:0905.2993) at press
[5] Baik J, Ben Arous G and Péché S 2005 Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices *Ann. Probab.* **33** 1643–97
[6] Baik J, Deift P and Johansson K 1999 On the distribution of the length of the longest increasing subsequence of random permutations *J. Am. Math. Soc.* **12** 1119–78
[7] Baik J, Ferrari P L and Péché S 2010 Limit process of stationary TASEP near the characteristic line *Commun. Pure Appl. Math.* **63** 1017–70
[8] Baik J, Kriecherbauer T, McLaughlin K T-R and Miller P D 2007 *Discrete Orthogonal Polynomials: Asymptotics and Applications (Annals of Mathematics Studies vol 164)* (Princeton, NJ: Princeton University Press)
[9] Baik J and Rains E M 2000 Limiting distributions for a polynuclear growth model with external sources *J. Stat. Phys.* **100** 523–41
[10] Baik J and Rains E M 2001 Symmetrized random permutations *Random Matrix Models and Their Applications (Math. Sci. Res. Inst. Publ. volume 40)* (Cambridge: Cambridge University Press) pp 1–19
[11] Baik J, Deift P and Suidan T 2010 *Some Combinatorial Problems and Random Matrix Theory* in preparation
[12] Balázs M, Komjáthy J and Seppäläinen T 2008 Microscopic concavity and fluctuation bounds in a class of deposition processes arXiv:0808.1177
[13] Balázs M, Quastel J and Seppäläinen T 2010 Scaling exponent for the Hopf–Cole solution of KPZ/stochastic Burgers arXiv:0909.4816
[14] Balázs M, Rassoul-Agha F and Seppäläinen T 2006 The random average process and random walk in a space-time random environment in one dimension *Commun. Math. Phys.* **266** 499–545
[15] Balázs M and Seppäläinen T 2010 Order of current variance and diffusivity in the asymmetric simple exclusion process *Ann. Math.* **171** 1237–65
[16] Balázs M and Seppäläinen T 2009 Fluctuation bounds for the asymmetric simple exclusion process *ALEA Lat. Am. J. Probab. Math. Stat.* **6** 1–24
[17] Bertini L and Giacomin G 1997 Stochastic Burgers and KPZ equations from particle systems *Commun. Math. Phys.* **183** 571–607
[18] Bertini L, Sole A De, Gabrielli D, Jona-Lasinio G and Landim C 2007 Stochastic interacting particle systems out of equilibrium *J. Stat. Mech.* P07014
[19] Binder P M, Paczuski M and Barma M 1994 Scaling of fluctuations in one-dimensional interface and hopping models *Phys. Rev.* E **49** 1174–81
[20] Blythe R A and Evans M R 2007 Nonequilibrium steady states of matrix-product form: a solver's guide *J. Phys. A: Math. Theor.* **40** R333–441

[21] Bornemann F 2009 On the numerical evaluation of distributions in random matrix theory: a review *Markov Proc. Rel. Fields* (arXiv:0904.1581) at press

[22] Bornemann F, Ferrari P L and Prähofer M 2008 The Airy_1 process is not the limit of the largest eigenvalue in GOE matrix diffusion *J. Stat. Phys.* **133** 405–15

[23] Borodin A and Ferrari P L 2008 Large time asymptotics of growth models on space-like paths: I. PushASEP *Electron. J. Probab.* **13** 1380–418

[24] Borodin A, Ferrari P L, Prähofer M and Sasamoto T 2007 Fluctuation properties of the TASEP with periodic initial configuration *J. Stat. Phys.* **129** 1055–80

[25] Borodin A, Ferrari P L and Sasamoto T 2008 Transition between Airy_1 and Airy_2 processes and TASEP fluctuations *Commun. Pure Appl. Math.* **61** 1603–29

[26] Bray A J 1994 Theory of phase-ordering kinetics *Adv. Phys.* **43** 357–459

[27] Calabrese P, Le Doussal P and Rosso A 2010 Free energy distribution of the directed polymer at high temperature *Europhys. Lett.* **90** 20002

[28] Cator E and Groeneboom P 2006 Second class particles and cube root asymptotics for Hammersley's process *Ann. Probab.* **34** 1273–95

[29] Chowdhury D, Santen L and Schadschneider A 2000 Statistical physics of vehicular traffic and some related systems *Phys. Rep.* **329** 199–329

[30] Cole J D 1951 On a quasi-linear parabolic equation occurring in aerodynamics *Q. Appl. Math.* **9** 225–36

[31] Corwin I, Ferrari P L and Péché S 2010 Limit processes for TASEP with shocks and rarefaction fans *J. Stat. Phys.* **140** 232–67

[32] Corwin I, Ferrari P L and Péché S 2010 Universality of slow decorrelation in KPZ growth arXiv:1001.5345

[33] Deift P 1999 *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach (Courant Lecture Notes in Mathematics vol 3)* (New York: Courant Institute of Mathematical Sciences)

[34] Deift P 2007 Universality for mathematical and physical systems *International Congress of Mathematicians* vol 1 (Zürich: European Mathematical Society) pp 125–52

[35] Deift P and Gioev D 2009 *Random Matrix Theory: Invariant Ensembles and Universality (Courant Lecture Notes in Mathematics vol 18)* (New York: Courant Institute of Mathematical Sciences)

[36] Deift P, Kriecherbauer T, McLaughlin K T-R, Venakides S and Zhou X 1999 Strong asymptotics of orthogonal polynomials with respect to exponential weights *Commun. Pure Appl. Math.* **52** 1491–552

[37] DeMasi A, Presutti E and Scacciatelli E 1989 The weakly asymmetric simple exclusion process *Ann. Inst. Henri Poincare* B **25** 1–38

[38] Derrida B 2007 Non-equilibrium steady states: fluctuations and large deviations of the density and of the current *J. Stat. Mech.* P07023

[39] Derrida B, Janowsky S A, Lebowitz J L and Speer E R 1993 Exact solution of the totally asymmetric exclusion process: shock profiles *J. Stat. Phys.* **73** 813–42

[40] Derrida B, Lebowitz J L, Speer E R and Spohn H 1991 Dynamics of an anchored Toom interface *J. Phys. A: Math. Gen.* **24** 4805–34

[41] Devillard P and Spohn H 1992 Universality class of interface growth with reflection symmetry *J. Stat. Phys.* **66** 1089–99

[42] Dotsenko V 2010 Bethe ansatz derivation of the Tracy–Widom distribution for one-dimensional directed polymers *Europhys. Lett.* **90** 20003

[43] Erdős L 2010 Universality of Wigner random matrices: a survey of recent results arXiv:1004.0861

[44] Evans M R and Hanney T 2005 Nonequilibrium statistical mechanics of the zero-range process and related models *J. Phys. A: Math. Gen.* **38** R195–239

[45] Ferrari P A 1992 Shock fluctuations in asymmetric simple exclusion *Probab. Theory Rel. Fields* **91** 81–101

[46] Ferrari P A and Fontes L R G 1998 Fluctuations of a surface submitted to a random average process *Electron. J. Probab.* **3** 1–34

[47] Ferrari P A, Kipnis C and Saada E 1991 Microscopic structure of travelling waves in the asymmetric simple exclusion process *Ann. Probab.* **19** 226–44

[48] Ferrari P L and Spohn H 2006 Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process *Commun. Math. Phys.* **265** 1–44

[49] Ferrari P L 2008 Slow decorrelations in Kardar–Parisi–Zhang growth *J. Stat. Mech.* P07022

[50] Ferrari P L 2008 The universal Airy_1 and Airy_2 processes in the totally asymmetric simple exclusion process *Integrable Systems and Random Matrices (Contemp. Math. vol 458)* (Providence, RI: American Mathematical Society) pp 321–32

[51] Forster D, Nelson D R and Stephen M J 1977 Large-distance and long-time properties of a randomly stirred fluid *Phys. Rev.* A **16** 732–49

[52] Frey E, Täuber U C and Hwa T 1996 Mode coupling and renormalization group results for the noisy Burgers equation *Phys. Rev.* E **53** 4424–38

[53] Golinelli O and Mallick K 2006 The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics *J. Phys. A: Math. Gen.* **39** 12679–705

[54] Hager J, Krug J, Popkov V and Schütz G M 2001 Minimal current phase and universal boundary layers in driven diffusive system *Phys. Rev.* E **63** 056110

[55] Halpin-Healy T and Zhang Y-C 1995 Kinetic roughening phenomena, stochastic growth, directed polymers and all that. Aspects of multidisciplinary statistical mechanics *Phys. Rep.* **254** 215–414

[56] Hopf E 1950 The partial differential equation $u_t + uu_x = \mu u_{xx}$ *Commun. Pure Appl. Math.* **3** 201–30

[57] Huse D A, Henley C L and Fisher D S 1985 Huse, Henley, and Fisher respond *Phys. Rev. Lett.* **55** 2924

[58] Imamura T and Sasamoto T 2004 Fluctuations of the one-dimensional polynuclear growth model with external sources *Nucl. Phys.* B **699** 503–44

[59] Johansson K 2000 Shape fluctuations and random matrices *Commun. Math. Phys.* **209** 437–76

[60] Johansson K 2002 Non-intersecting paths, random tilings and random matrices *Probab. Theory Relat. Fields* **123** 225–80

[61] Kallabis H and Krug J 1999 Persistence of Kardar–Parisi–Zhang interfaces *Europhys. Lett.* **45** 20–5

[62] Kardar M 1987 Replica Bethe ansatz studies of two-dimensional interfaces with quenched random impurities *Nucl. Phys.* B **290** 582–602

[63] Kardar M, Parisi G and Zhang Y-C 1986 Dynamic scaling of growing interfaces *Phys. Rev. Lett.* **56** 889–92

[64] Kardar M and Zhang Y-C 1987 Scaling of directed polymers in random media *Phys. Rev. Lett.* **58** 2087–90

[65] Kelly F 1979 *Reversibility and Stochastic Networks* (New York: Wiley)

[66] Kipnis C and Landim C 1999 *Scaling Limits of Interacting Particles Systems* (Berlin: Springer)

[67] Knuth D E 1970 Permutations, matrices, and generalized Young tableaux *Pac. J. Math* **34** 709–27

[68] König W 2005 Orthogonal polynomial ensembles in probability theory *Probab. Surv.* **2** 385–447

[69] Krech M 1997 Short-time scaling behavior of growing interfaces *Phys. Rev.* E **55** 668–79

[70] Krim J and Palasantzas G 1995 Experimental observations of self-affine scaling and kinetic roughening at submicron lengthscales *Int. J. Mod. Phys.* B **9** 599–632

[71] Krug J 1991 Boundary-induced phase transitions in driven diffusive systems *Phys. Rev. Lett.* **67** 1881–5

[72] Krug J 1997 Origins of scale invariance in growth processes *Adv. Phys.* **46** 139–282

[73] Krug J 2000 Phase separation in disordered exclusion models *Braz. J. Phys.* **30** 97–104

[74] Krug J and García J 2000 Asymmetric particle systems on $\mathbb{R}$ *J. Stat. Phys.* **99** 31–55

[75] Krug J and Halpin-Healy T 1998 Ground-state energy anisotropy for directed polymers in random media *J. Phys. A: Math Gen.* **31** 5939–52

[76] Krug J, Meakin P and Halpin-Healy T 1992 Amplitude universality for driven interfaces and directed polymers in random media *Phys. Rev.* A **45** 638–53

[77] Krug J and Spohn H 1988 Universality classes for deterministic surface growth *Phys. Rev.* A **38** 4271–83

[78] Krug J and Spohn H 1989 Anomalous fluctuations in the driven and damped sine-Gordon chain *Europhys. Lett.* **8** 219–24

[79] Krug J and Spohn H 1991 Kinetic roughening of growing surfaces *Solids Far from Equilibrium* ed C Godrèche (Cambridge: Cambridge University Press)

[80] Krug J and Tang L-H 1994 Disorder induced unbinding in confined geometries *Phys. Rev.* E **50** 104

[81] Kuijlaars A B J, McLaughlin K T-R, Assche W Van and Vanlessen M 2004 The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$ *Adv. Math.* **188** 337–98

[82] Kuijlaars A B J and Vanlessen M 2003 Universality for eigenvalue correlations at the origin of the spectrum *Commun. Math. Phys.* **243** 163–91

[83] Lebowitz J L, Presutti E and Spohn H 1988 Microscopic models of hydrodynamic behavior *J. Stat. Phys.* **51** 841–62

[84] Levin E and Lubinsky D S 2009 On the Airy reproducing kernel, sampling series, and quadrature formula *Integral Eqns Operator Theory* **63** 427–38

[85] Liggett T M 1999 *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* (Berlin: Springer)

[86] Lubinsky D S 2008 Universality limits in the bulk for arbitrary measures on compact sets *J. Anal. Math.* **106** 373–94

[87] Lubinsky D S 2009 Universality limits for random matrices and de Branges spaces of entire functions *J. Funct. Anal.* **256** 3688–729

[88] Majumdar S N 2007 Random matrices, the Ulam problem, directed polymers and growth models and sequence matching *Complex Systems* ed J-P Bouchaud, M Mézard and J Dalibard (Amsterdam: Elsevier)

[89] Majumdar S N and Nechaev S 2004 Anisotropic ballistic deposition model with links to the Ulam problem and the Tracy–Widom distribution *Phys. Rev.* E **69** 011103

[90] Maunuksela J, Myllys M, Kähkönen O-P, Timonen J, Provatas N, Alava M J and Ala Nissila T 1997 Kinetic roughening in slow combustion of paper *Phys. Rev. Lett.* **79** 1515–8
[91] McLaughlin K T-R and Miller P D 2008 The $\overline{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights *Int. Math. Res. Not.* **2008** 075
[92] Meakin P, Ramanlal P, Sander L M and Ball R C 1986 Ballistic deposition on surfaces *Phys. Rev.* A **34** 5091–103
[93] Mehta M L 2004 *Random Matrices* (*Pure and Applied Mathematics* vol 142) 3rd edn (Amsterdam: Elsevier/Academic)
[94] Michely T and Krug J 2004 *Islands, Mounds and Atoms. Patterns and Processes in Crystal Growth Far from Equilibrium (Springer Series in Surface Science vol 42)* (Heidelberg: Springer)
[95] Miettinen L, Myllys M, Merikoski J and Timonen J 2005 Experimental determination of KPZ height-fluctuation distributions *Eur. Phys. J.* B **46** 55–60
[96] Nagao T and Sasamoto T 2004 Asymmetric simple exclusion process and modified random matrix ensembles *Nucl. Phys.* B **699** 487–502
[97] Plischke M and Bergersen B 2006 *Equilibrium Statistical Physics* (Singapore: World Scientific)
[98] Plischke M, Rácz Z and Liu D 1987 Time-reversal invariance and universality of two-dimensional growth models *Phys. Rev.* B **35** 3485–95
[99] Prähofer M and Spohn H 2000 Statistical self-similarity of one-dimensional growth processes *Physica* A **279** 342–52
[100] Prähofer M and Spohn H 2000 Universal distributions for growth processes in 1+1 dimensions and random matrices *Phys. Rev. Lett.* **84** 4882–5
[101] Prähofer M and Spohn H 2002 Current fluctuations for the totally asymmetric simple exclusion process *In and Out of Equilibrium: Probability with a Physics Flavor* ed Vladas Sidoravicius (Berlin: Springer)
[102] Prähofer M and Spohn H 2002 Scale invariance of the PNG droplet and the Airy process *J. Stat. Phys.* **108** 1071–106
[103] Quastel J and Valkó B 2007 $t^{1/3}$ Superdiffusivity of finite-range asymmetric exclusion processes on $\mathbb{Z}$ *Commun. Math. Phys.* **273** 379–94
[104] Quastel J and Valkó B 2008 A note on the diffusivity of finite-range asymmetric exclusion processes on $\mathbb{Z}$ *In and out of Equilibrium: 2 (Progr. Probab. vol 60)* (Basel: Birkhäuser) pp 543–49
[105] Rákos A and Schütz G M 2005 Current distribution and random matrix ensembles for an integrable asymmetric fragmentation process *J. Stat. Phys.* **118** 511–30
[106] Resnick S I 2002 *Adventures in Stochastic Processes* (Boston, MA: Birkhäuser)
[107] Rost H 1981 Non-equilibrium behaviour of a many particle process: density profile and local equilibria *Probab. Theory Rel. Fields* **58** 41–53
[108] Sagan B E 2001 *The Symmetric Group (Graduate Texts in Mathematics vol 203)* 2nd edn (New York: Springer)
[109] Sasamoto T 2005 Spatial correlations of the 1D KPZ surface on a flat substrate *J. Phys. A: Math. Gen.* **38** L549–56
[110] Sasamoto T 2007 Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques *J. Stat. Mech.* P07007
[111] Sasamoto T and Spohn H 2009 Superdiffusivity of the 1D lattice Kardar–Parisi–Zhang equation *J. Stat. Phys.* **137** 917–35
[112] Sasamoto T and Spohn H 2010 The crossover regime for the weakly asymmetric simple exclusion process *J. Stat. Phys.* **140** 209–31
[113] Sasamoto T and Spohn H 2010 Exact height distributions for the KPZ equation with narrow wedge initial condition *Nucl. Phys.* B **834** 523–42
[114] Sasamoto T and Spohn H 2010 One-dimensional Kardar–Parisi–Zhang equation: an exact solution and its universality *Phys. Rev. Lett.* **104** 230602
[115] Schadschneider A and Schreckenberg M 1993 Cellular-automaton models and traffic flow *J. Phys. A: Math Gen.* **26** L679–83
[116] Schmittmann B and Zia R K P 1995 *Statistical Mechanics of Driven Diffusive Systems* (London: Academic)
[117] Schreckenberg M, Schadschneider A, Nagel K and Ito N 1995 Discrete stochastic models for traffic flow *Phys. Rev.* E **51** 2939
[118] Schütz G M 1997 Exact solution of the master equation for the asymmetric exclusion process *J. Stat. Phys.* **88** 427–45
[119] Schütz G M, Ramaswamy R and Barma M 1996 Pairwise balance and invariant measures for generalized exclusion processes *J. Phys. A: Math Gen.* **29** 837–43
[120] Seppäläinen T 1999 Existence of hydrodynamics for the totally asymmetric simple k-exclusion process *Ann. Probab.* **27** 361–415

[121] Singha S B 2005 Persistence of surface fluctuations in radially growing surfaces *J. Stat. Mech.* P08006
[122] Spitzer F 1970 Interaction of Markov processes *Adv. Math.* **5** 246–90
[123] Spohn H 1983 Long range correlations for stochastic lattice gases in a non-equilibrium steady state *J. Phys. A: Math Gen.* **16** 4275
[124] Spohn H 1991 *Large Scale Dynamics of Interacting Particles* (Berlin: Springer)
[125] Spohn H 2006 Exact solutions for KPZ-type growth processes, random matrices, and equilibrium shapes of crystals *Physica* A **369** 71–99
[126] Takeuchi K A and Sano M 2010 Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals *Phys. Rev. Lett.* **104** 230601
[127] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel *Commun. Math. Phys.* **159** 151–74
[128] Tracy C A and Widom H 1998 Correlation functions, cluster functions, and spacing distributions for random matrices *J. Stat. Phys.* **92** 809–35
[129] Tracy C A and Widom H 2002 Distribution functions for largest eigenvalues and their applications *Proc. Int. Congress of Mathematicians* (Beijing: Beijing Higher Ed. Press) vol 1 pp 587–96
[130] Tracy C A and Widom H 2008 A Fredholm determinant representation in ASEP *J. Stat. Phys.* **132** 291–300
[131] Tracy C A and Widom H 2008 Integral formulas for the asymmetric simple exclusion process *Commun. Math. Phys.* **279** 815–44
[132] Tracy C A and Widom H 2009 Asymptotics in ASEP with step initial condition *Commun. Math. Phys.* **290** 129–54
[133] Tracy C A and Widom H 2009 On ASEP with step Bernoulli initial condition *J. Stat. Phys.* **137** 825–38
[134] Tracy C A and Widom H 2009 On the distribution of a second-class particle in the asymmetric simple exclusion process *J. Phys. A: Math. Theor.* **42** 425002, 6
[135] Tracy C A and Widom H 2009 Total current fluctuations in the asymmetric simple exclusion process *J. Math. Phys.* **50** 095204, 4
[136] Tracy C A and Widom H 2010 Formulas for ASEP with two-sided Bernoulli initial condition *J. Stat. Phys.* **140** 619–34
[137] van Beijeren H, Kutner R and Spohn H 1985 Excess noise for driven diffusive systems *Phys. Rev. Lett.* **54** 2026–9
[138] van Kampen N G 2001 *Stochastic Processes in Physics and Chemistry* (Amsterdam: Elsevier)
[139] van Moerbeke P 2002 Random matrices and permutations, matrix integrals and integrable systems *(Séminaire Bourbaki vol 1999/2000) Astérisque* 411–33
[140] van Moerbeke P 2008 Nonintersecting Brownian motions, integrable systems and orthogonal polynomials *Probability, Geometry and Integrable Systems (Math. Sci. Res. Inst. Publ. vol 55)* (Cambridge: Cambridge Univ. Press) pp 373–96
[141] Vollmer J 2002 Chaos, spatial extension, transport, and non-equilibrium thermodynamics *Phys. Rep.* **372** 131–267
[142] Wolfram S 1983 Statistical mechanics of cellular automata *Rev. Mod. Phys.* **55** 601
[143] Yaguchi H 1986 Stationary measures for an exclusion process on one-dimensional lattices with infinitely many hopping sites *Hiroshima Math. J.* **16** 449–75
[144] Zia R K P and Schmittmann B 2007 Probability currents as principal characteristics in the statistical mechanics of non-equilibrium steady states *J. Stat. Mech.* P07012