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7. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 16: Growth shapes, Legendre transforms and the Wulff construction

We consider a one-dimensional interface described by a height function $h(x,t)$. On the macroscopic scale the evolution equation reads

$$\frac{\partial h}{\partial t} = V\left(\frac{\partial h}{\partial x}\right) \tag{1}$$

where $V(u)$ denotes the inclination-dependent growth velocity. We are interested in 'droplet' solutions of the form $h(x,t) = tg(x/t)$. Inserting this ansatz into (1) yields the differential equation

$$g(z) - zg'(z) = V(g'(z)). \tag{2}$$

a) Assume that the second derivative of V has a definite sign, $V''(u) > 0$ or $V''(u) < 0$ everywhere. Then show that the solution of (2) is

$$g(z) = \min_u [V(u) + uz] \text{ if } V'' > 0 \tag{3}$$

$$g(z) = \max_u [V(u) + uz] \text{ if } V'' < 0 \tag{4}$$

and in both cases $g''(z)V''(u(z)) = -1$, where $u(z)$ is the value of u at which the extremum on the right hand side of (3,4) occurs.

b) Apply (4) to the discrete time, deterministic version of the single step model (the growth model equivalent of CA 184, see Exercises 10 and 12). Interpret the result in terms of the microscopic dynamics.

c) For the polynuclear growth model (to be defined later in the lectures) the inclination-dependent growth velocity has the form

$$V(u) = c\sqrt{u_0^2 + u^2} \tag{5}$$

with constants $c > 0$ and u_0. Compute the corresponding growth shape $g(z)$ and sketch the functions $V(u)$ and $g(z)$.

d) Instead of describing the interface shape in terms of the height function $g(z)$, we may introduce a polar representation $r(\varphi)$ through

$$g(z) = r(\varphi)\cos\varphi, \quad z = r(\varphi)\sin(\varphi). \tag{6}$$

Similarly we introduce the *normal* growth velocity $V_n(\vartheta)$ through

$$V_n(\vartheta) = V(-\tan\vartheta)\cos\vartheta. \tag{7}$$

The angles φ and ϑ are defined on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and are measured relative to the h-axis. Show that in this representation the Legendre transform (3) is equivalent to the *Wulff construction*[1]

$$r(\varphi) = \min_{\vartheta} \frac{V_n(\vartheta)}{\cos(\varphi - \vartheta)}. \tag{8}$$

Exercise 17: Minimal energy paths on the square lattice

It was shown in the lectures that the single step model can be mapped onto an optimization problem for directed paths in a random energy landscape. The key object of interest is the ground state energy

$$E(i,j) = \min_{\omega} E(\omega) = \min_{\omega} \sum_{(k,l)\in\omega} \epsilon_{kl} \tag{9}$$

where the $\epsilon_{kl} < 0$ are exponentially distributed random variables with mean $\langle \epsilon_{kl} \rangle = -1$, and the minimum is taken over all directed paths ending at (i,j).

a) For paths ending on the diagonal the mapping yields the result

$$\langle E(n,n) \rangle = -4n = 2\langle E(\omega) \rangle \tag{10}$$

where $\langle E(\omega) \rangle = -2n$ is the mean energy of such paths (note that each path passes through $2n$ sites). How does this compare to what you would get if all the $\binom{2n}{n}$ paths ending at (n,n) were independent?
Hint: You may assume that, for large n, the path energies are Gaussian random variables with mean $-2n$ and variance $2n$. The expected value of the minimum of N independent Gaussian random variables of mean μ and variance σ^2 is $\mu - \sigma\sqrt{2 \ln N}$. Finally, use Stirling's approximation to evaluate $\binom{2n}{n}$ for large n.

b) Now we generalize the result (10) to arbitrary endpoints. To be specific, compute the *ground state energy per unit length*

$$e(\phi) = \lim_{i,j\to\infty} \frac{E(i,j)}{i+j} \tag{11}$$

as a function of the angle ϕ that the mean path direction encloses with the diagonal, $\phi = \arctan[\frac{j-i}{j+i}]$.
Hint: Use the known growth shape $h(x,t) = tg(x/t) = \frac{1}{2}[1 + (x/t)^2]$ together with the relations $h = i + j$, $x = i - j$ to derive a quadratic equation for e.

[1] G. Wulff, Z. Kristallogr. Mineral. *34* (1901) 449.