$q \neq \frac{1}{2}$: Evolution of $\langle \eta_x \rangle$ is coupled to $\langle \eta_x \eta_{x\pm} \rangle$ which in turn is coupled to higher order correlations

$\Rightarrow$ infinite hierarchy, no closed set of equations.

Note: Although product measure

$$\langle \eta_x \eta_{x'} \rangle = \langle \eta_x \rangle \langle \eta_{x'} \rangle$$

holds in the stationary state at constant density, this is not true for an evolving, spatially inhomogeneous density profile.

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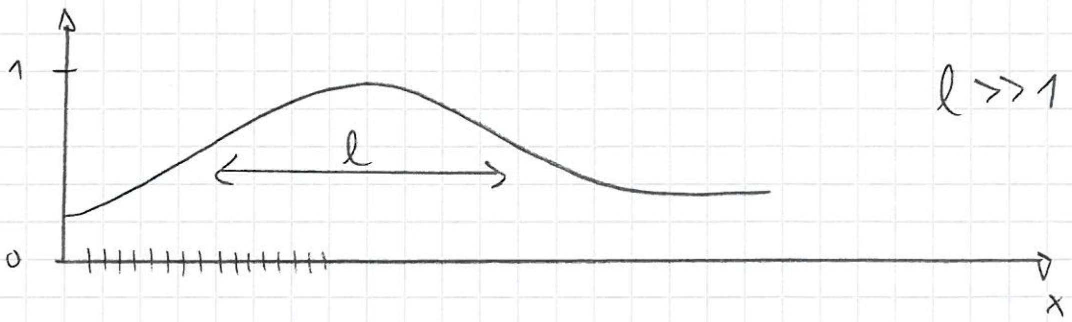
b) The hydrodynamic limit

Question: Start ASEP from Bernoulli measure with a slowly varying density $\rho(x, 0)$:



(i) Does the stochastic time evolution stay close to a Bernoulli measure with a time-dep. density profile $\rho(x,t)$?

(ii) Is this true on average (over many runs) or even for single configurations ?

(iii) What is the evolution equation for $\rho(x,t)$?

Heuristically, (i) and (ii) are true (for simple configurations) for $l \to \infty$, because the process converges locally to the stationary state with density $\rho(x,t)$.

To derive the evolution equation, we note that because of particle conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} j(x,t) = 0$$ for some local current $j(x,t)$

and for $l \to \infty$ (by the same argument)

$$j(x,t) \longrightarrow J(\rho(x,t))$$

$$\Rightarrow \text{(E)} \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} J(\rho) = 0$$ hyperbolic conservation law, Euler equation

For rigorous results see e.g. C. Kipnis, C. Landim, Scaling limits of interacting particle systems (Springer, 1999).

Note: (E) holds also for cases where the stationary measure is not Bernoulli (and may not even be explicitly known).

c) Method of characteristics

To solve the Euler equation (E), rewrite it as

$$\frac{d\rho}{dt} = -\frac{\partial}{dx} J(\rho) = \underline{-c(\rho)\frac{d\rho}{dx}} \quad , \quad c = \frac{dJ}{d\rho}$$

If c were constant, the initial density profile $\rho_0(x)$ would simply translate at speed c:

$$\rho(x,t) = \rho(x-ct, 0) =: \rho_0(x-ct)$$

In the general case, points of density ρ travel along characteristics at the kinematic wave speed $c(\rho)$. To show this, introduce

$$\underline{\xi(x_0, t) = x_0 + c(\rho_0(x_0))\cdot t}$$

to denote the position of a point of density $\rho_0(x_0)$ initially located at x_0. We assume that the relation between ξ and x_0 can be inverted, which is certainly true at $t=0$, and denote the inverse function by $x_0(\xi, t)$. Then the solution of (E) with initial condition ρ_0 is given by

$$\underline{\rho(x,t) = \rho_0(x_0(x,t))} \qquad (C)$$

Proof: $x_0(x,t) = x - c(\rho_0(x_0(x,t)))\cdot t$

$$\Rightarrow \frac{dx_0}{dx} = 1 - c'\rho_0' \frac{dx_0}{dx}\cdot t$$

$$\Rightarrow \frac{dx_0}{dx} = (1 + c'\rho_0' t)^{-1}$$

$$\frac{\partial x_0}{\partial t} = -c - c'\rho_0' \frac{\partial x_0}{\partial t}\, t \quad \Rightarrow \quad \frac{\partial x_0}{\partial t} = -c\, \frac{\partial x_0}{\partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \rho_0(x_0(x,t)) = \rho_0' \frac{\partial x_0}{\partial t} = -c\rho_0' \frac{\partial x_0}{\partial x} =$$

$$= -c \frac{\partial}{\partial x} \rho_0(x_0(x,t)) \quad \square$$

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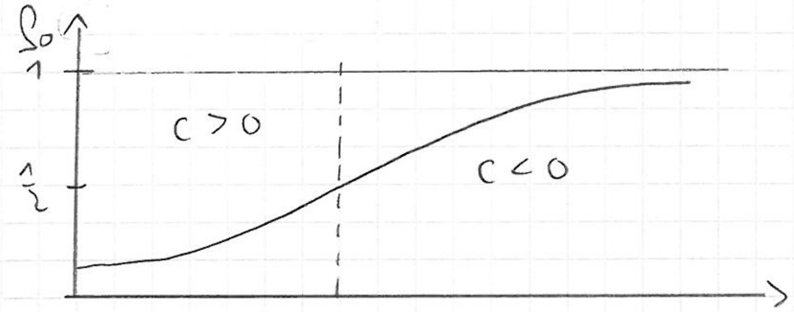
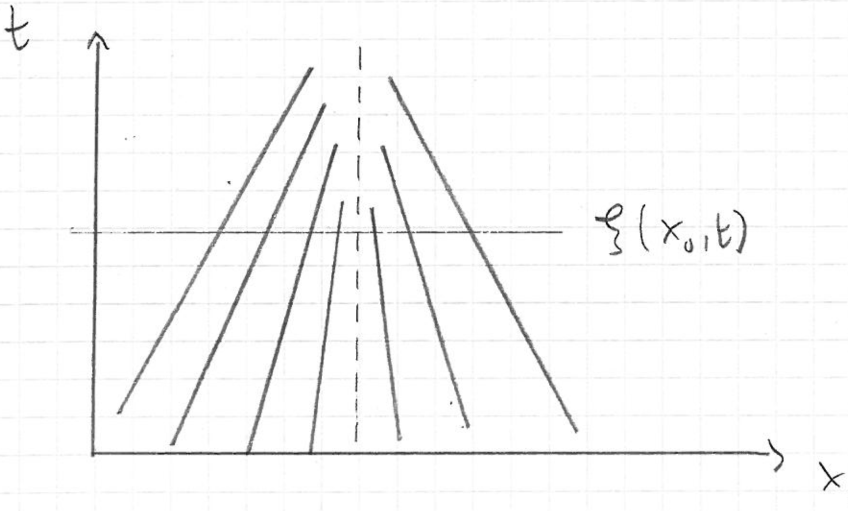
d) Shock formation

Consider the TASEP with an increasing initial profile ($\rho_0' > 0$).



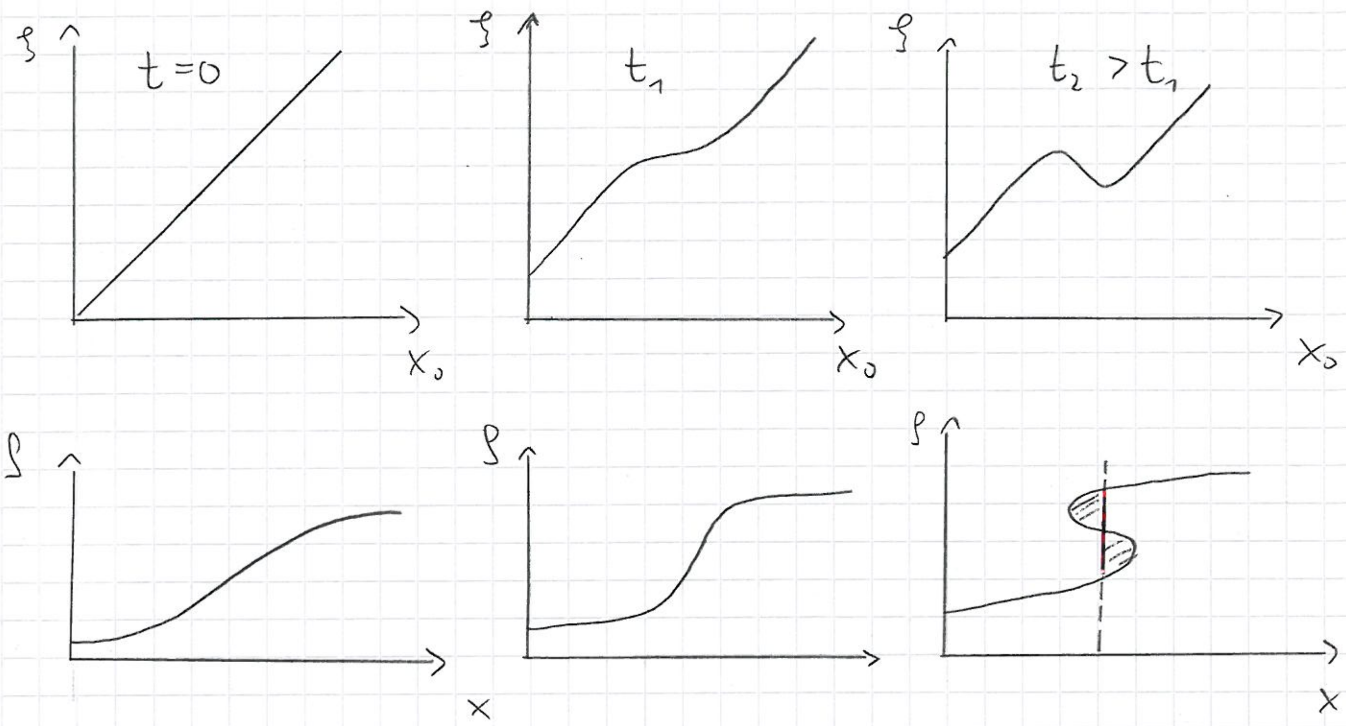
$$c(\rho) = 1 - 2\rho$$



$\Rightarrow$ characteristics converge and will collide in finite time. At this point the mapping

$$x_0 \longrightarrow \xi(x_0,t)$$

ceases to be invertible, and the density profile defined by (C) becomes multivalued:



Physically the profile develops a shock discontinuity. The position of the shock is determined by mass conservation (cf. Maxwell construction). Alternatively one may consider viscosity solutions, i.e. solutions of

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} J(\rho) = \nu \frac{\partial^2 \rho}{\partial x^2} \quad \text{with } \nu \to 0.$$

- When does the shock appear?

(i) $$\frac{\partial \xi}{\partial x_0} = 1 + c' \rho_0' t \overset{!}{=} 0$$

$$\Rightarrow t^* = -\frac{1}{c' \rho_0'}$$

(iv) Local analysis: Expand $j(\rho)$ to second order around $\bar{\rho}$:

$$\left.\begin{aligned} j(\bar{\rho}+u) &= j(\bar{\rho}) + c(\bar{\rho})\, u + \frac{1}{2}\lambda(\bar{\rho})\, u^2 \\ \lambda(\bar{\rho}) &= \frac{d^2 j}{d\rho^2}(\bar{\rho}) = c'(\bar{\rho}) \end{aligned}\right\}$$

$$\Rightarrow \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} j = -c(\bar{\rho})\frac{\partial u}{\partial x} - \lambda(\bar{\rho})\, u \frac{\partial u}{\partial x}$$

Galilei-transformation to moving frame:

$$u(x,t) = \tilde{u}(x - c(\bar{\rho})t,\, t)$$

$$\Rightarrow \underline{\frac{\partial \tilde{u}}{\partial t} = -\lambda \tilde{u} \frac{\partial \tilde{u}}{\partial x}}$$ "inviscid Burgers eq."

Locally the density profile is linear

$$\Rightarrow \text{ansatz} \quad \tilde{u}(x,t) = a(t)\, x$$

$$\Rightarrow \dot{a}\, x = -\lambda a^2 x \Rightarrow \underline{\dot{a} = -\lambda a^2}$$

$$\Rightarrow \underline{a(t) = \frac{a(0)}{1 + \lambda a(0) t}}$$

For the TASEP $\lambda = -2$

$$\Rightarrow \underline{a(t) = \frac{a(0)}{1 - 2a(0)t}}.$$

Two cases:

- $a(0) = \frac{\partial s}{\partial x}(0,0) > 0 \Rightarrow$ finite time blowup at

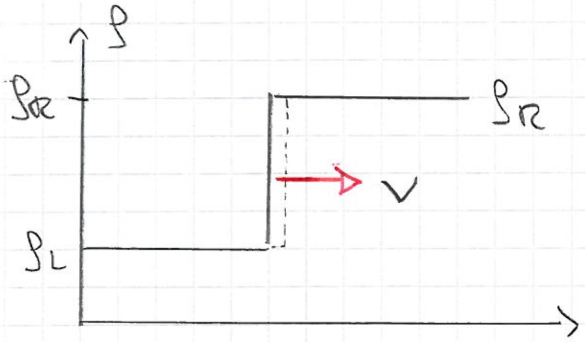
$$t^* = \frac{1}{2a(0)} = -\frac{1}{c'\rho_0'}$$

$\Rightarrow$ first shock forms at the point of maximal initial density gradient

- $a(0) < 0 \Rightarrow$ profile flattens out as

$$a(t) = \frac{a(0)}{1+2|a(0)|t} \sim \frac{1}{t}$$

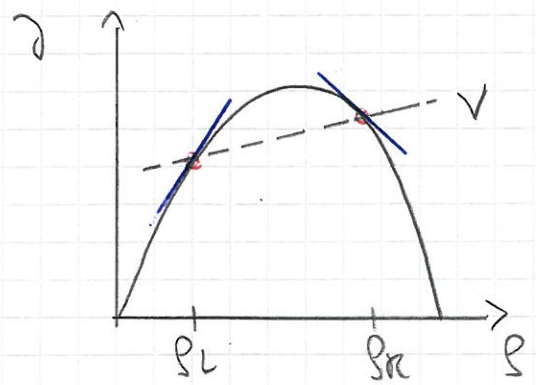
- Shock motion cannot be predicted from the Euler equation but is determined through mass conservation:



$$J(\rho_R) - J(\rho_L) = V(\rho_R - \rho_L)$$

$$\Rightarrow V = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L}$$

- Shock stability is related to the convexity of $J(\rho)$:



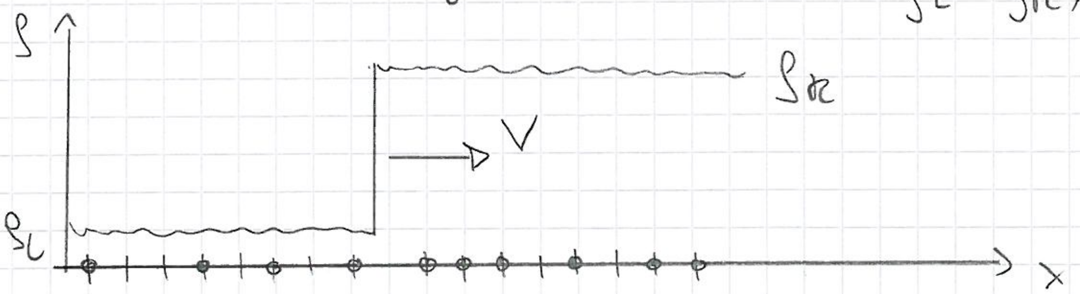
$$c(\rho_L) > V > c(\rho_R)$$

Density perturbations travel at speed c and are therefore absorbed at the shock.

Microscopic structure of the shock:

In addition to the spatially uniform stationary distribution on the finite ring, the ASEP on $\mathbb{Z}$ possesses shock measures with

$$\{\eta_x\} \sim \begin{cases} \text{Bernoulli of density } \rho_R, & x \to \infty \\ \quad -\text{''}- \qquad\qquad\quad \rho_L < \rho_R, & x \to -\infty \end{cases}$$



How to locate the shock position microscopically?

Use "second class particles":

Second class particles travel at speed c and therefore accumulate at the shock position.

In a reference frame moving with the 2nd class particle, the shock is microscopically sharp on a length scale

$$l_{\text{shock}} \sim (\rho_R - \rho_L)^{-2}$$

(Derrida et al., 1993)