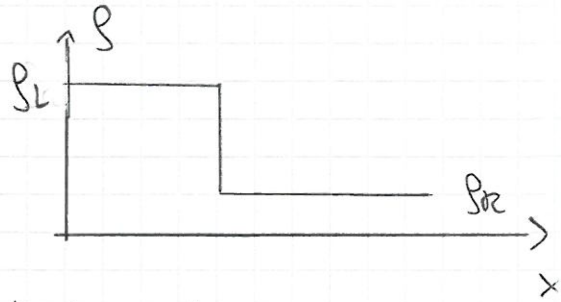
e) Rarefaction waves

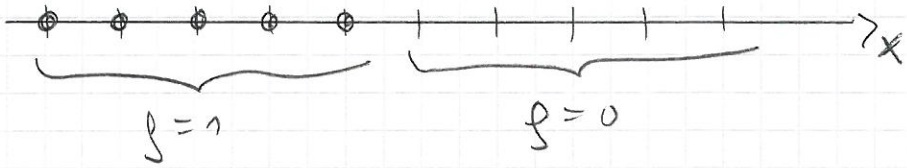
How does a "inverse shock"

$$\rho(x,0) = \begin{cases} \rho_L & x < 0 \\ \rho_R < \rho_L & x > 0 \end{cases}$$

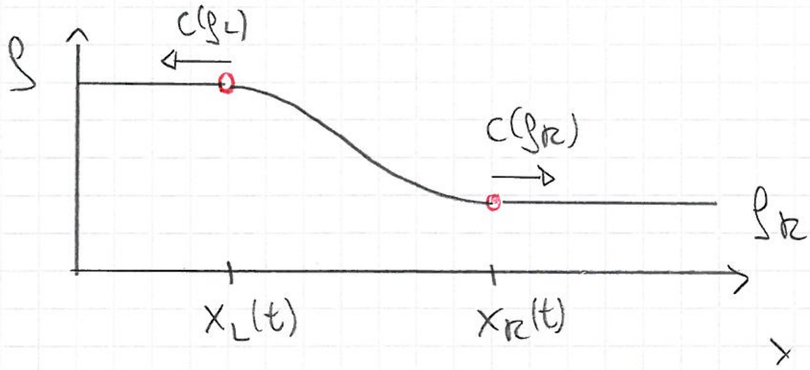


evolve? In this case characteristics diverge.

Example: Dissolution of a "traffic jam" in the ASEP



Expect:



$$\rho(x,t) = \begin{cases} \rho_L & x < x_L(t) = c(\rho_L)\cdot t \\ \rho_R & x > x_R(t) = c(\rho_R)\cdot t \\ \phi(x/t) & x_L(t) < x < x_R(t) \end{cases}$$

$\Rightarrow$ For $x_L < x < x_R$ this implies

$$\frac{\partial \rho}{\partial t} = -\frac{x}{t^2}\,\phi' \quad , \quad \frac{\partial \rho}{\partial x} = \frac{1}{t}\,\phi'$$

$\Rightarrow$ with $\xi = x/t$ we have

$$-\frac{1}{t}\,\xi\,\phi'(\xi) + c(\phi(\xi))\,\frac{1}{t}\,\phi'(\xi) = 0$$

$$\Rightarrow \quad c(\phi(\xi)) = \xi, \qquad \underline{\phi(\xi) = c^{-1}(\xi)}$$

For the TASEP with $c = 1-2\rho$ this implies

$$\underline{\phi(\xi) = \tfrac{1}{2}(1-\xi)}$$

linear profile

5° Equivalent problems

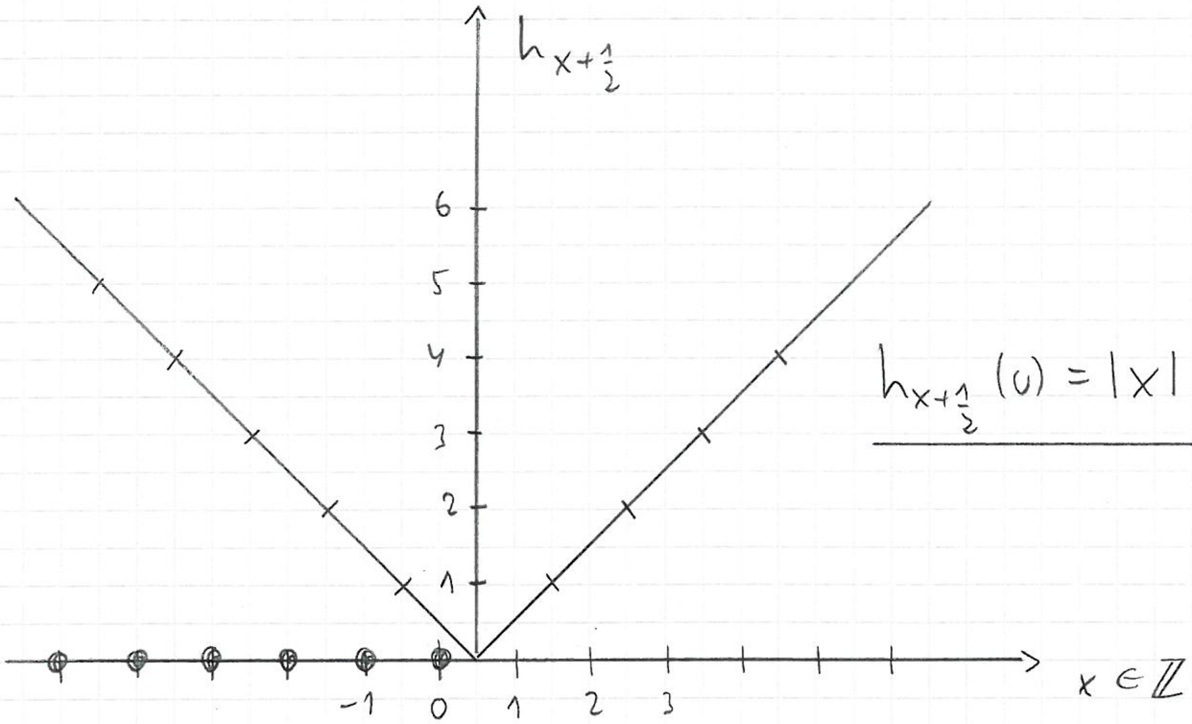
a) <u>The corner growth model</u> (H. Rost, 1981)

Consider the TASEP on $\mathbb{Z}$ with an "inverse shock" (= step) initial condition:

$$\eta_x = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

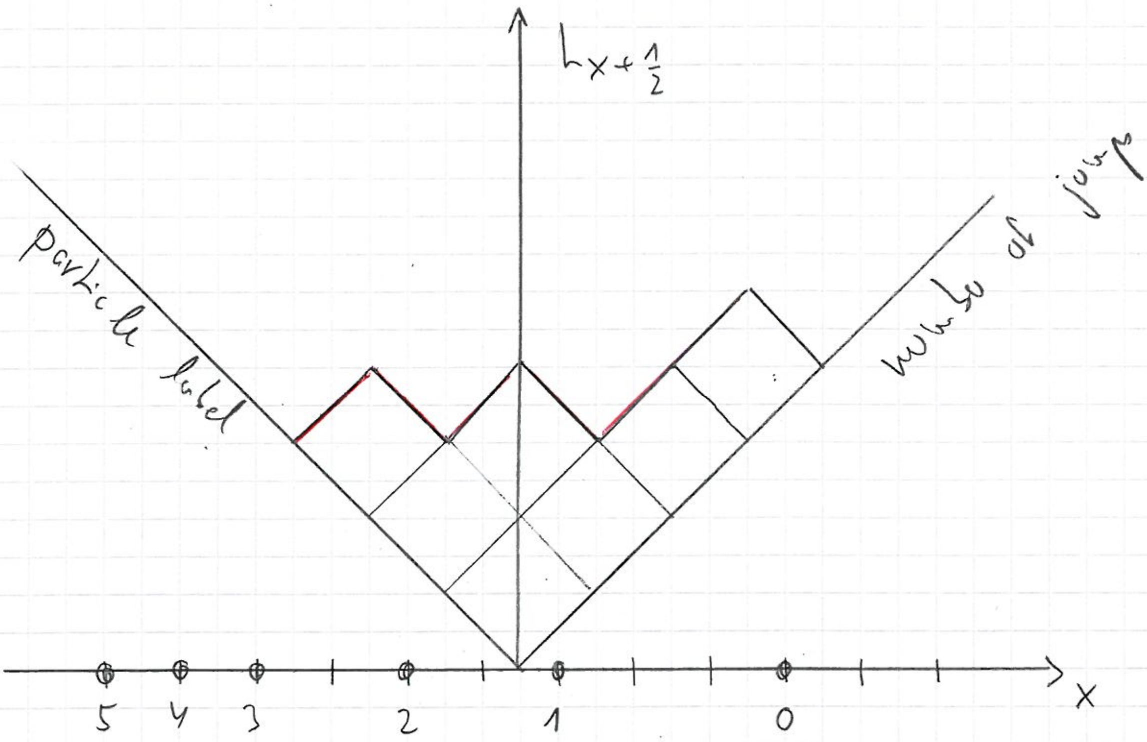
We now assign a height configuration living on the bonds of the TASEP lattice:

$$\begin{cases} h_{x+\frac{1}{2}} - h_{x-\frac{1}{2}} = 1 - 2\eta_x = \begin{cases} 1 & \eta_x = 0 \\ -1 & \eta_x = 1 \end{cases} \\ h_{\frac{1}{2}} = 0 \end{cases}$$



$$h_{x+\frac{1}{2}}(0) = |x|$$

$\Downarrow$ a few jumps



Remarks:

- A jump $x \to x+1$ increases the corresponding height

$$h_{x+\frac{1}{2}} \to h_{x+\frac{1}{2}} + 2$$

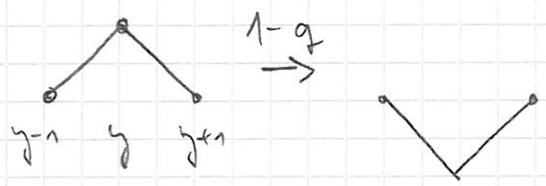
$\Rightarrow$ $\frac{1}{2}\left(h_{x+\frac{1}{2}}(t) - h_{x+\frac{1}{2}}(0)\right)$ is the number of particles that have jumped from x to $x+1$ up to time t (= local current)

- The number of jumps performed by a given particle can also be read off from the height configuration.
- Exclusion interaction translates into the single step condition $|h_y - h_{y+1}| = 1$ which is preserved by the growth rule

$$h_y \to h_y + 2 \quad \text{if} \quad \underbrace{h_{y\pm 1} = h_y + 1}_{y \text{ is local minimum}}$$

- Backward jumps correspond to the evaporation of particles from local maxima:

$$h_y \to h_y - 2 \quad \text{if} \quad h_{y\pm 1} = h_y - 1$$



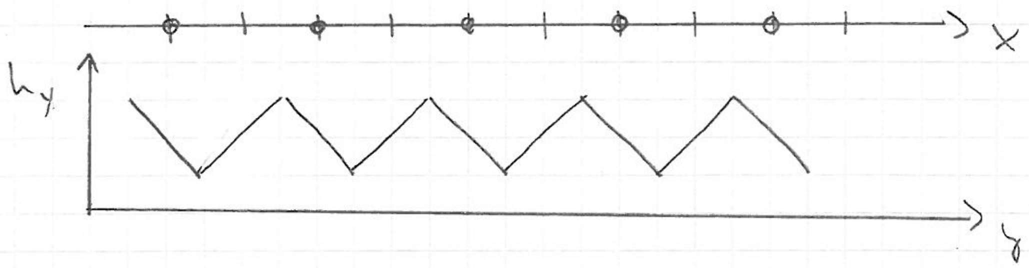
- With reference to Sect. 3°, we note that

$$W_{\circ\bullet}(\eta) = \#\{\text{local maxima of } h_y\} =$$

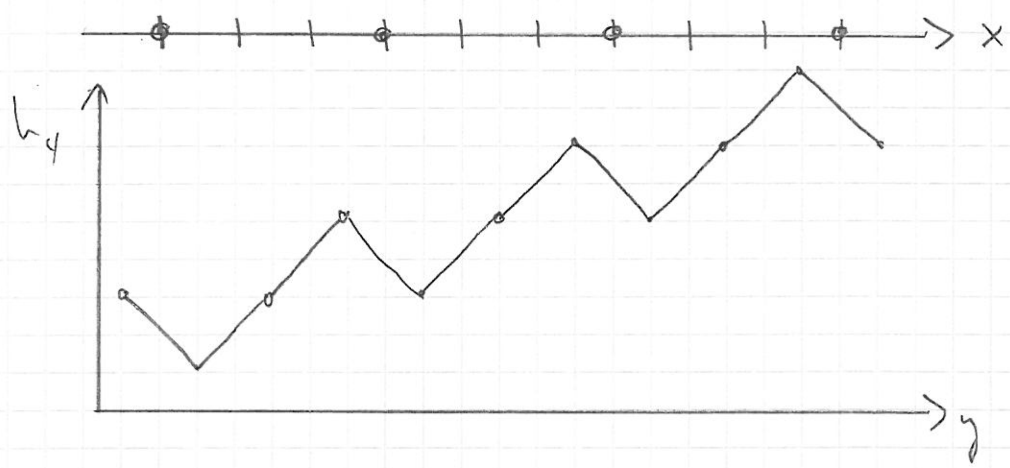
$$= \#\{\text{local minima}\} = W_{\bullet\circ}(\eta)$$

Other initial conditions

(i) flat / deterministic: $\eta_x = \frac{1}{2}\left(1+(-1)^x\right)$

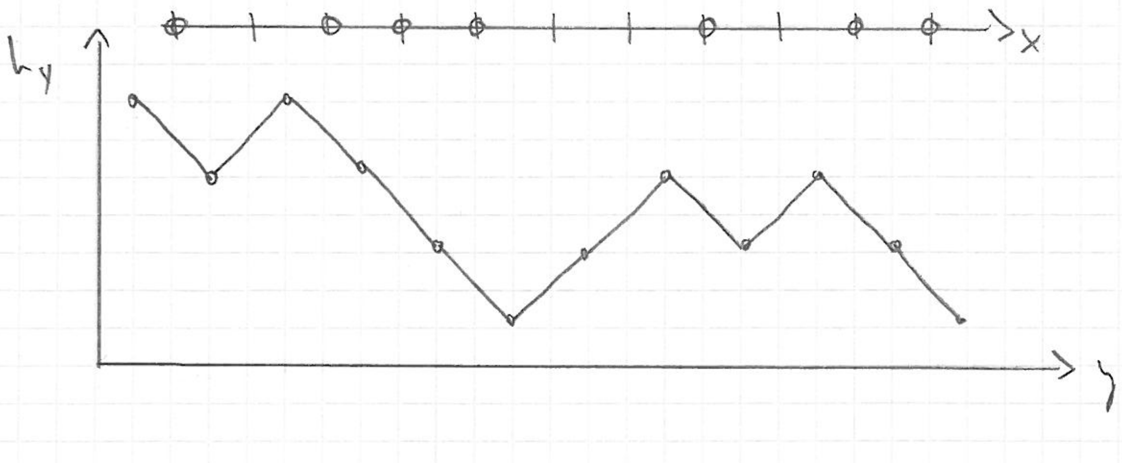


(ii) tilted / deterministic: Every n-th site occupied



$\Rightarrow$ slope $\underline{u = \langle h_{y+1} - h_y \rangle = 1 - 2\rho}$

(iii) rough / random: Bernoulli measure for $\{\eta_x\}$



To characterize the roughness we introduce the height difference correlation function

$$G(r) = \langle (h_{y+r} - h_y)^2 \rangle - \langle (h_{y+r} - h_y) \rangle^2$$

$$= \underline{4\rho(1-\rho)|r|} \quad \text{using} \quad \langle \eta_i \eta_j \rangle = \rho^2 + \rho(1-\rho)\delta_{ij}$$

$\Rightarrow$ interface performs a random walk in y with "diffusion constant" $4\rho(1-\rho)$.

Hydrodynamics of interface motion

The mapping implies

$$\begin{cases} \text{particle current } J(\rho) \longrightarrow \text{growth velocity } \dfrac{\partial h}{\partial t} \\ \text{particle density } \rho \longrightarrow \text{interface slope } \dfrac{\partial h}{\partial y} \end{cases}$$

$\Rightarrow$ the macroscopic evolution of the interface is governed by

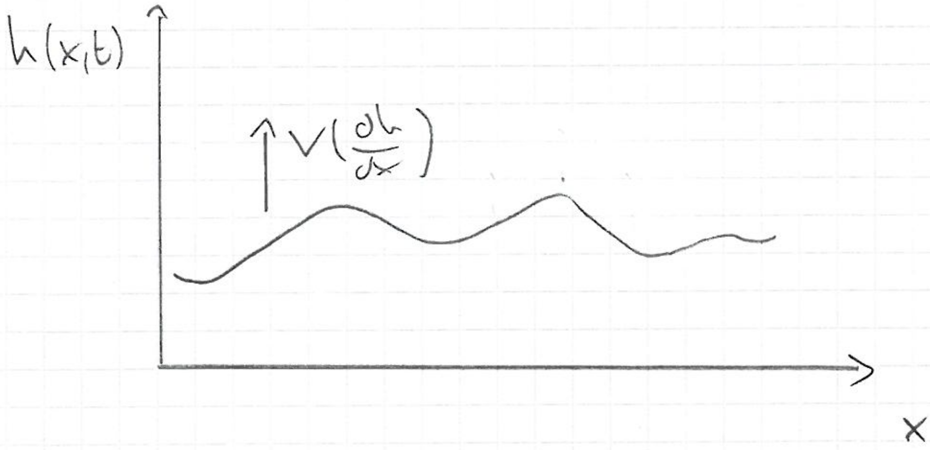
$$\underline{\frac{\partial h}{\partial t} = V\left(\frac{\partial h}{\partial y}\right)}$$

"Hamilton-Jacobi":

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}\right) = 0$$

with $V(u) \hat{=} J(\rho(u))$

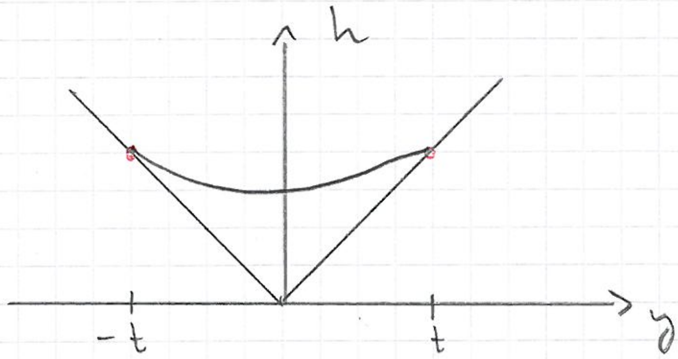
"inclination-dependent growth velocity"



To be specific, in the present case

$$\gamma(\rho) = \rho(1-\rho), \quad u = 1-2\rho, \quad V = 2\gamma$$

$$\Rightarrow \quad \underline{V(u) = \tfrac{1}{2}(1-u^2)}$$



Applications:

(i) Corner growth

Ansatz: $h(y,t) = t\, g(y/t)$

$$\Rightarrow \quad \frac{\partial h}{\partial t} = g - \frac{y}{t} g', \quad \frac{\partial h}{\partial y} = g'$$

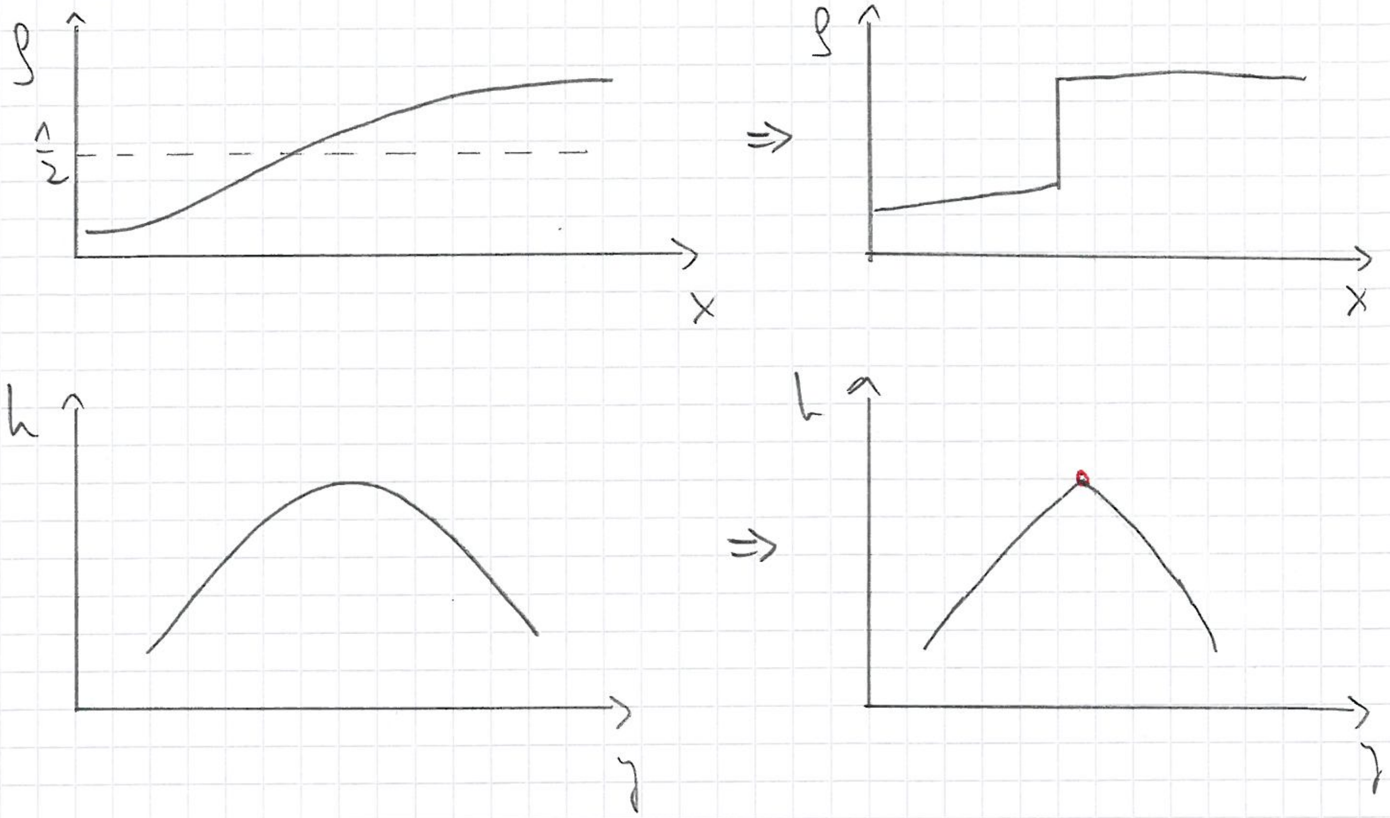
$$\Rightarrow \quad g(z) - z g'(z) = V(g'(z)) = \tfrac{1}{2}(1 - g'(z)^2)$$

Legendre transform

$$\Rightarrow \quad \underline{g(z) = \tfrac{1}{2}(1+z^2)}$$

→ Problem

(ii) Shock formation



⇒ interface forms cusp singularity in finite time.

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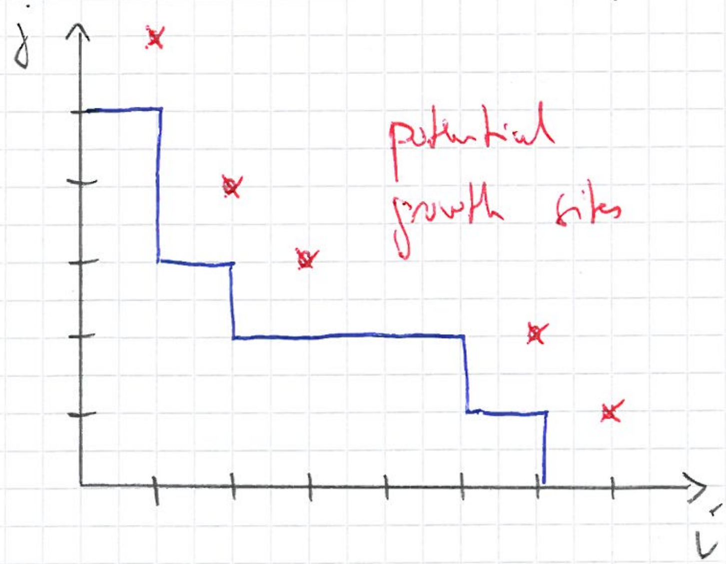
b) The waiting time representation

Consider the corner growth geometry:



define

$t(i,j) :=$ time at which the interface reaches (i,j)