The "single step" constraint implies that site (i,j) can be filled only after sites $(i-1,j)$ and $(i,j-1)$:

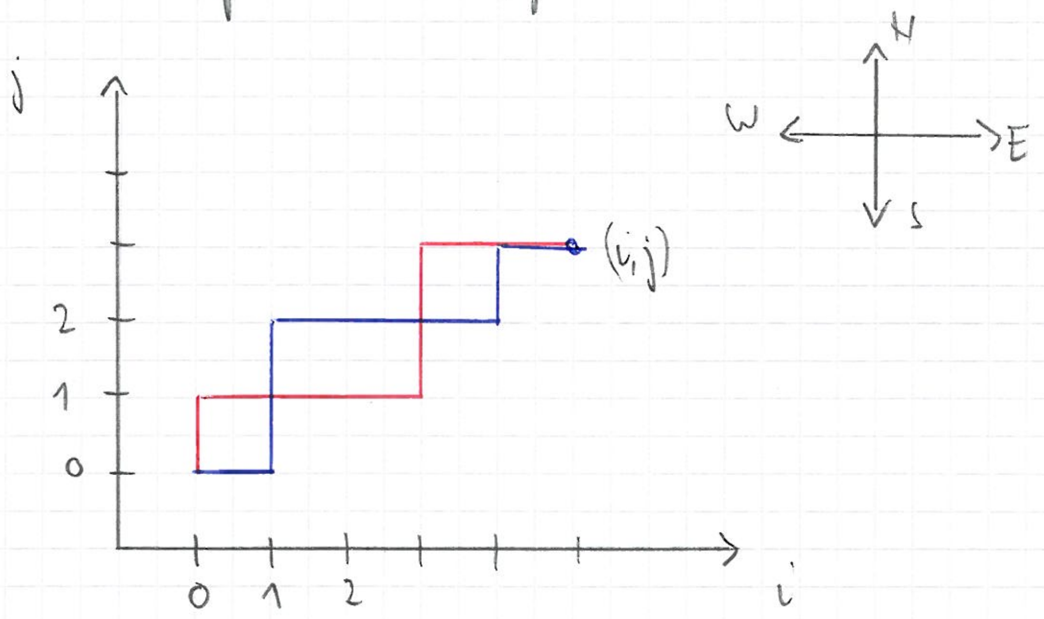
$$\circledast \quad t(i,j) = \max\left[\, t(i-1,j),\, t(i,j-1)\right] + \tau_{ij}$$

where the τ_{ij} are exponentially distributed random waiting times with $\langle \tau_{ij} \rangle = 1$

At the boundaries $(i=1,\ j=1)$ we have instead

$$\left.\begin{aligned} t(i,1) &= t(i-1,1) + \tau_{i,1} \\ t(1,j) &= t(1,j-1) + \tau_{1,j} \end{aligned}\right\}$$

To formulate a solution to $\circledast$ we introduce the set of north-east directed paths ω from the origin to (i,j):



- length of paths $i+j$
- number of paths $\binom{i+j}{i}$ $\Big\}$

To each path ω we associate the waiting time

$$t(\omega) = \sum_{(i,j)\in\omega} \tau_{ij}$$

then the $t(i,j)$ are given by

$$t(i,j) = \max_{\substack{\omega \text{ ending} \\ \text{at } (i,j)}} \left(t(\omega) \right)$$

Proof: By iteration.

Remarks:

- In mathematics this problem is known as last passage percolation.
- Defining "site energies" $\varepsilon_{ij} = -\tau_{ij}$, and "path energies" $E(\omega) = -t(\omega)$

  then $E(i,j)$ is the ground state energy of a directed polymer moving in the random medium defined by the ε_{ij}:

$$E(i,j) = \min_{\substack{\omega \text{ ending} \\ \text{at } (i,j)}} \left[\sum_{(i,j)\in\omega} \varepsilon_{ij} \right]$$

"DPRM" (Kardar, Zhang 1987)

- The DPRM is a problem of extreme value theory for highly correlated random variables.

Example: Consider paths along the diagonal, $(i,j) = (n,n)$

$$\Rightarrow \quad \langle E(\omega) \rangle = -2n$$

On the other hand we know from hydrodynamics that the growth speed along the diagonal is

$$v(0) = \frac{1}{2} \quad \Rightarrow \quad \left.\begin{array}{l}\text{height } h \text{ is reached} \\ \text{at time } t = 2h\end{array}\right\}$$

and since $h = 2n$ we have

$$E(n,n) = \min_{\omega} \{E(\omega)\} = -4n = 2\langle E \rangle$$

Problems:
- Compare to i.i.d. statistics
- Compute ground state energy as fct. of path direction

- The same approach can be applied to the dTASEP. In this case the waiting times τ_{ij} are integer valued with a geometric distribution:

$$P(\tau) = \pi(1-\pi)^{\tau-1}, \quad \tau \geq 1$$

6° Fluctuation theory

Goal: Develop a theory for density fluctuations in the ASEP & height fluctuations (roughness) in the single step model.

three levels

Idea: Langevin approach: hydrodynamics + noise.

a) Linear density fluctuations

Recall: The hydrodynamic equation for the symmetric exclusion process is linear diffusion

$$\frac{\partial \rho}{\partial t} = \nu \frac{\partial^2 \rho}{\partial x^2} \quad , \quad \nu = \frac{1}{2} \Gamma_0 a^2$$

We consider fluctuations ϕ around a homogeneous density $\bar{\rho}$: $\rho(x,t) = \bar{\rho} + \phi(x,t)$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial}{\partial x} \xi(x,t) \qquad (*)$$

where $\xi(x,t)$ is a fluctuating current modeled as white noise in space and time

$$\langle \xi(x,t) \rangle = 0 \, , \quad \langle \xi(x,t)\, \xi(x',t') \rangle = D\, \delta(x-x')\, \delta(t-t')$$

D: noise strength (not diffusion coefficient)

Remark: White noise is always to be understood as a limit of a correlated process:

$$\left.\begin{aligned}\langle \xi(x,t)\,\xi(x',t')\rangle &= a_x^{-1} a_t^{-1}\, \mathcal{C}\left(\frac{x-x'}{a_x}, \frac{t-t'}{a_t}\right)\\ &\downarrow \quad a_x, a_t \to 0\\ &D\,\delta(x-x')\,\delta(t-t')\end{aligned}\right\}$$

To solve (*) we introduce spatial Fourier transforms:

$$\left.\begin{aligned}\hat{\phi}(k,t) &= \int dx\, e^{ikx}\, \phi(x,t)\\ \hat{\xi}(k,t) &= \int dx\, e^{ikx}\, \xi(x,t)\end{aligned}\right\}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\hat{\phi}(k,t) = -\nu k^2 \hat{\phi}(k,t) + ik\,\hat{\xi}(k,t)$$

Ornstein-Uhlenbeck process with friction νk^2 and force $ik\,\hat{\xi}(k,t)$.

Solution:

$$\hat{\phi}(k,t) = e^{-\nu k^2 t}\,\hat{\phi}(k,0) + $$

$$+ \, ik\int_0^t ds\, e^{-\nu k^2(t-s)}\,\hat{\xi}(k,s)$$

For simplicity we set $\hat{\phi}(k,0) = 0$.

$$\Rightarrow \langle \hat{\phi}(k,t)\, \hat{\phi}(k',t') \rangle =$$

$$= -kk' \int_0^t ds \int_0^{t'} ds'\; e^{-\nu k^2 (t-s)}\, e^{-\nu k'^2 (t'-s')} \times$$

$$\times \langle \hat{\xi}(k,s)\, \hat{\xi}(k',s') \rangle$$

Now $\langle \hat{\xi}(k,t)\, \hat{\xi}(k',t') \rangle =$

$$= \int dx \int dx'\; e^{ikx}\, e^{ik'x'} \underbrace{\langle \xi(x,t)\, \xi(x',t') \rangle}_{= D\,\delta(x-x')\,\delta(t-t')}$$

$$= D\,\delta(t-t') \underbrace{\int dx\; e^{i(k+k')x}}_{= 2\pi\,\delta(k+k')} = 2\pi D\,\delta(t-t')\,\delta(k+k')$$

$$\Rightarrow \langle \hat{\phi}(k,t)\, \hat{\phi}(k',t') \rangle =$$

$$= 2\pi D k^2\, \delta(k+k') \int_0^t ds \int_0^{t'} ds'\; e^{-\nu k^2 (t+t'-s-s')} \times \delta(s-s')$$

$$= 2\pi D k^2\, \delta(k+k')\, e^{-\nu k^2 (t+t')} \int_0^{\min(t,t')} ds\; e^{2\nu k^2 s}$$

Take $t > t' \Rightarrow \min(t,t') = t'$

$$\Rightarrow \langle \hat{\phi}(k,t)\, \hat{\phi}(k',t') \rangle =$$

$$= 2\pi D k^2 \delta(k+k')\, e^{-\nu k^2 (t+t')} \frac{1}{2\nu k^2} \left(e^{2\nu k^2 t'} - 1 \right)$$

$$= \pi \left(\frac{D}{\nu}\right) \delta(k+k') \left(e^{-\nu k^2 |t-t'|} - e^{-\nu k^2 (t+t')} \right)$$

In the stationary state $(t, t' \to \infty)$ this reduces to

$$\langle \hat{\phi}(k,t)\, \hat{\phi}(k',t') \rangle = \pi \left(\frac{D}{\nu}\right) \delta(k+k')\, e^{-\nu k^2 |t-t'|}$$

$$\Rightarrow \langle \phi(x,t)\, \phi(x',t') \rangle =$$

$$= \frac{1}{4\pi} \frac{D}{\nu} \int dk\, e^{-ik(x-x')}\, e^{-\nu k^2 |t-t'|}$$

$$= \frac{1}{4\sqrt{\pi}} \frac{D}{\nu^{3/2}} \cdot \frac{1}{|t-t'|^{1/2}}\, e^{-\frac{(x-x')^2}{4\nu|t-t'|}}$$

$\Rightarrow$ density fluctuations are correlated on the scale of the diffusion length $\sim \sqrt{\nu |t-t'|}$.

$\underline{t \to t'}$: $\langle \phi(x,t)\, \phi(x',t) \rangle = \dfrac{D}{2\nu} \delta(x-x')$

which can be used to determine the noise strength D.

To this end consider the fluctuation in particle number in a region of size L:

$$N = \int_0^L dx\, \rho(x,t) = \bar{\rho} L + \int_0^L dx\, \phi(x,t)$$

$$\Rightarrow \langle (N - \langle N \rangle)^2 \rangle = \int_0^L dx \int_0^L dx' \langle \phi(x,t)\, \phi(x',t) \rangle =$$

$$= \frac{D}{2\nu} \cdot L \overset{!}{=} \rho(1-\rho) \cdot L \quad \Rightarrow \quad \underline{\frac{D}{2\nu} = \rho(1-\rho)}$$

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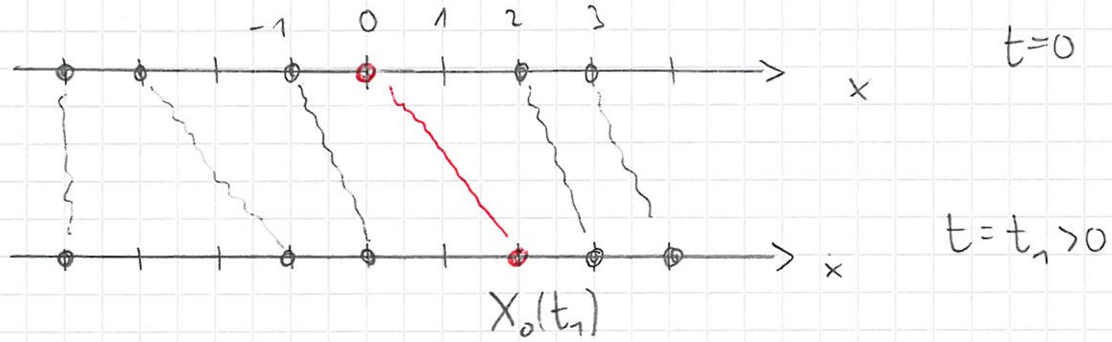
b) Single file diffusion

Consider the motion of a tagged particle originally located at $x = 0$:



$\Rightarrow$ up to corrections of $O(1)$ $X_0(t)$ is given by the time-integrated particle current through the origin:

$$\underline{X_0(t) = \frac{1}{\rho} \int_0^t j(0,s)\, ds}$$