$$\Rightarrow \quad \langle X_0(t)^2 \rangle = \frac{1}{\rho^2} \int_0^t ds \int_0^t ds' \, \langle j(0,s)\, j(0,s') \rangle$$

To determine the current correlations we use the continuity equation:

$$\frac{\partial \phi}{\partial t} + \frac{\partial j}{\partial x} = 0$$

$$\Big\downarrow \text{ Fourier transform}$$

$$\frac{\partial}{\partial t} \hat{\phi}(k,t) = -ik\, \hat{j}(k,t)$$

$$\Rightarrow \langle \hat{j}(k,t)\, \hat{j}(k',t') \rangle = -\frac{1}{kk'} \frac{\partial}{\partial t} \frac{\partial}{\partial t'} \langle \hat{\phi}(k,t)\, \hat{\phi}(k',t') \rangle$$

$$= + \pi \rho(1-\rho)\, \delta(k+k') \frac{1}{k^2} \frac{\partial}{\partial t} \frac{\partial}{\partial t'} e^{-\nu k^2 |t-t'|} =$$

$$= - \pi \rho(1-\rho)\, \delta(k+k') \frac{1}{k^2} \frac{d^2}{d\tau^2} e^{-\nu k^2 |\tau|} \qquad (\tau = t-t')$$

Now $\frac{d}{d\tau} e^{-\nu k^2 |\tau|} = -\nu k^2 \operatorname{sgn}(\tau)\, e^{-\nu k^2 |\tau|}$

$$\left. \begin{aligned} \frac{d^2}{d\tau^2} e^{-\nu k^2 |\tau|} &= \nu^2 k^4 e^{-\nu k^2 |\tau|} - \\ &\quad - 2\nu k^2 \delta(\tau)\, e^{-\nu k^2 |\tau|} \end{aligned} \right\}$$

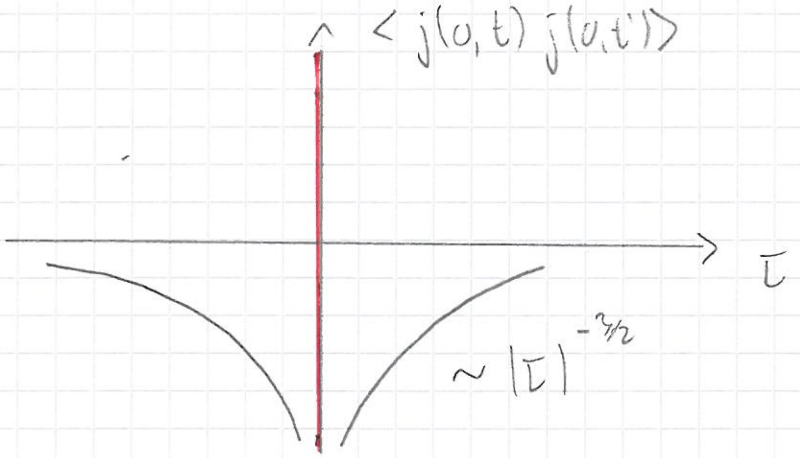
$$\Rightarrow \langle \hat{j}(k,t)\, \hat{j}(k',t')\rangle = \left[-\pi v^2 k^2 + 2\pi v\, \delta(\tau)\right] \times$$

$$\times\, \rho(1-\rho)\, \delta(k+k')\, e^{-v k^2 |\tau|}$$

$$\Rightarrow \langle j(0,t)\, j(0,t')\rangle = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' \langle \hat{j}(k,t)\, \hat{j}(k',t')\rangle$$

$$= \frac{1}{4\pi}\, v \rho(1-\rho) \int_{-\infty}^{\infty} dk\; e^{-vk^2|\tau|} \left[2\delta(\tau) - v k^2\right]$$

$$= \frac{1}{4\pi}\, v\, \rho(1-\rho) \left[\sqrt{\frac{\pi}{v|\tau|}}\; 2\delta(\tau) - \frac{1}{2}\sqrt{\frac{\pi}{v|\tau|^3}}\right] =: C_j(\tau)$$



The quantity of interest is

$$\langle X_0(t)^2\rangle = \frac{1}{\rho^2} \int_0^t ds \int_0^t ds' \langle j(0,s)\, j(0,s')\rangle =$$

$$= \frac{2}{\rho^2} \int_0^t ds \int_0^s d\tau\; C_j(\tau)$$

$$\Rightarrow \quad \frac{d^2}{dt^2} \langle X_0(t)^2 \rangle = \frac{2}{\rho^2} C_j(t)$$

<u>Ansatz:</u> $\langle X_0(t)^2 \rangle = a \cdot |t|^{1/2}$

$$\Rightarrow \frac{d}{dt} \langle X_0(t)^2 \rangle = \frac{1}{2} a \, \mathrm{sgn}(t) \, |t|^{-1/2}$$

$$\frac{d^2}{dt^2} \langle X_0(t)^2 \rangle = a \, \delta(t) \, |t|^{-\frac{1}{2}} - \frac{1}{4} a |t|^{-3/2}$$

$$\overset{!}{=} \frac{2}{\rho^2} \cdot \frac{1}{4} \sqrt{\frac{\nu}{\pi}} \, \rho(1-\rho) \left[2 \delta(t) |t|^{-1/2} - \frac{1}{2} |t|^{-3/2} \right]$$

$$\Rightarrow \quad \underline{a = \frac{1-\rho}{\rho} \sqrt{\frac{\nu}{\pi}}}$$

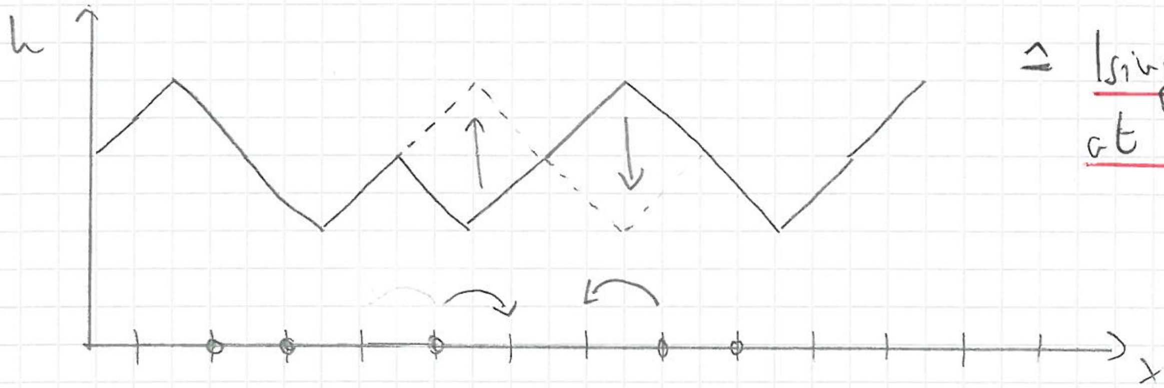
<u>Limits:</u>

$$\left.\begin{array}{lll} \rho \to 1 & \Rightarrow & a \to 0, \text{ no diffusion} \\ \rho \to 0 & \Rightarrow & a \to \infty, \text{ normal diffusion} \end{array}\right\}$$

<u>Remark:</u> $X_0(t)$ is a realization of fractional Brownian motion with Hurst exponent $H = 1/4$. (subdiffusive)

c) The Edwards-Wilkinson equation

Consider the interface representation of the symmetric exclusion process:



$\hat{=}$ Ising interface at $T=0$

$\Rightarrow$ describes fluctuations of a one-dimensional interface in equilibrium.

Continuum description:

$$\frac{\partial h}{\partial x} = 1 - 2\rho = 1 - 2\bar{\rho} - 2\phi$$

$$\frac{\partial h}{\partial t} = 2j = 2\left(-\nu \frac{\partial \phi}{\partial x} + \xi(x,t)\right)$$

$$= \nu \frac{\partial^2 h}{\partial x^2} + 2\xi(x,t)$$

one-dimensional Edwards-Wilkinson eq. (1982)

This is identical to the equation for ϕ, but the noise is not conserved ($\partial_x \xi \to \xi$).

Height correlations:

We subtract the mean slope $1-2\bar{\rho}$

$$h(x,t) \longrightarrow \tilde{h}(x,t) = h(x,t) - (1-2\bar{\rho})x$$

$$\Rightarrow \frac{\partial \tilde{h}}{\partial x} = -2\phi$$ and drop the tilde

$$\Rightarrow \hat{h}(k,t) = \frac{2}{ik}\hat{\phi}(k,t)$$

$$\Rightarrow \langle \hat{h}(k,t)\hat{h}(k',t')\rangle = 4\pi\left(\frac{D}{\nu}\right)\frac{\delta(k+k')}{k^2}e^{-\nu k^2|t-t'|}$$

in the stationary state

$$\Rightarrow \langle h(x,t)h(x',t')\rangle = \frac{1}{\pi}\left(\frac{D}{\nu}\right)\int dk\,\frac{1}{k^2}e^{-ik(x-x')} \times e^{-\nu k^2|t-t'|}$$

$= \infty$ because of the divergence at $k=0$

$\Rightarrow$ interface position is delocalized by long-wavelength fluctuation

$\Rightarrow$ roughening

To describe fluctuations of rough interfaces we recall the height difference correlation function (Sect. 5° a))

$$G(x-x', t-t') = \langle (h(x,t) - h(x',t'))^2 \rangle =$$

$$= 2(\langle h(x,t)^2 \rangle - \langle h(x,t)\, h(x',t') \rangle) =$$

$$= \frac{2}{\pi}\left(\frac{D}{\nu}\right) \int dk \, \frac{1}{k^2} \left(1 - e^{-ik(x-x')} e^{-\nu k^2 |t-t'|}\right) =$$

$$= \frac{4}{\pi}\left(\frac{D}{\nu}\right) \int_0^\infty \frac{dk}{k^2} \left(1 - \cos k(x-x')\, e^{-\nu k^2 |t-t'|}\right)$$

Special cases:

(i) $t = t'$: $G(x-x', 0) =$

$$= \frac{4}{\pi}\left(\frac{D}{\nu}\right) \int_0^\infty \frac{dk}{k^2} \left(1 - \cos k(x-x')\right) = \quad \Big| \; q = k(x-x')$$

$$= \frac{4}{\pi}\left(\frac{D}{\nu}\right) |x-x'| \underbrace{\int_0^\infty \frac{dq}{q^2} (1-\cos q)}_{=\pi/2} = 2\left(\frac{D}{\nu}\right) |x-x'|$$

and with $\frac{D}{2\nu} = \bar{\rho}(1-\bar{\rho})$ we recover the result of the "microscopic" calculation in Sect. 5° a).

(ii) $\underline{x = x'}$: $G(0, t-t') = \underline{\langle (h(x,t) - h(x,t'))^2 \rangle}$

$$= \frac{4}{\pi}\left(\frac{D}{\nu}\right) \int_0^\infty \frac{dk}{k^2} \left(1 - e^{-\nu k^2 |t-t'|}\right) \quad \Big|_{q = k(\nu|t-t'|)^{1/2}}$$

$$= \frac{4}{\pi} \frac{D}{\sqrt{\nu}} |t-t'|^{1/2} \underbrace{\int_0^\infty \frac{dq}{q^2} \left(1 - e^{-q^2}\right)}_{= \sqrt{\pi}} \underline{\sim |t-t'|^{1/2}}$$

$$\sim X_0^2$$

because $h(x,t)$ also measures the integrated particle current (Sect. 5° a)). Details: Problems

For later reference we note that the full correlation function has scaling form:

$\underline{G(x-x', t-t')} =$

$$= \frac{4}{\pi} \frac{D}{\nu} \int \frac{dk}{k^2} \left(1 - \cos \underbrace{k(x-x')}_{q} e^{-\nu k^2 |t-t'|}\right) =$$

$$= \frac{4}{\pi}\frac{D}{\nu}|x-x'| \underbrace{\int_0^\infty \frac{dq}{q^2}\left(1-\cos q\, e^{-\nu q^2 |t-t'|/(x-x')^2}\right)}_{=:\ \mathcal{G}\left(|t-t'|/(x-x')^2\right)}$$

$$\sim |x-x'|^{2\alpha}\, \mathcal{G}\left(|t-t'|/|x-x'|^{z}\right)$$

$\alpha = \frac{1}{2}$ roughness exponent

$z = 2$ dynamic exponent

Special cases:

(i) $t = t'$: $\mathcal{G}(0) = \text{const.}$, $G(x-x', 0) \sim |x-x'|^{2\alpha}$

(ii) $x = x'$: To obtain a finite expression for $x \to x'$, the scaling function $\mathcal{G}$ has to diverge as

$$\mathcal{G}(\tau) \sim \tau^{2\alpha/z} \quad \text{for} \quad \tau = \frac{t-t'}{|x-x'|^z} \to \infty$$

$$\Rightarrow \langle (h(x,t)-h(x,t'))^2 \rangle \sim |t-t'|^{2\beta}$$

with $\beta = \frac{\alpha}{z} = \frac{1}{4}$.