7° Kardar-Parisi-Zhang theory

Goal: Develop a mesoscopic fluctuation theory for the ASEP & single step model.

a) Stochastic Burgers equation

We start from the hydrodynamic eq. (Sect. 4°)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} J(\rho) = 0$$

and expand around a uniform state of density $\bar{\rho}$ (cf. 4° d)): $\rho = \bar{\rho} + u$

$$\Rightarrow \quad J(\rho) = J(\bar{\rho}) + \underbrace{J'(\bar{\rho})}_{= c(\bar{\rho})} u + \frac{1}{2} \underbrace{J''(\bar{\rho})}_{\lambda(\bar{\rho})} u^2$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} J = - c(\bar{\rho}) \frac{\partial u}{\partial x} - \lambda(\bar{\rho}) u \frac{\partial u}{\partial x}$$

The linear term is eliminated by a Galilei transformation: $u(x,t) = \tilde{u}(x - c(\bar{\rho}) t, t)$

$$\Rightarrow \quad \frac{\partial \tilde{u}}{\partial t} = - \lambda \tilde{u} \frac{\partial \tilde{u}}{\partial x} \qquad \text{inviscid Burgers}$$

We know that this equation develops discontinuities (shocks) in finite time; adding white noise would therefore lead to an ill-defined problem

⇒ regularize by "viscosity" ν (see Exercise 13) (and rename $\tilde{u} \to u$):

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - \lambda u \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \xi(x,t)$$

$$\langle \xi(x,t)\, \xi(x',t')\rangle = D\, \delta(x-x')\, \delta(t-t')$$

stochastic Burgers equation

Remarks:

- Introduced 1977 by Forster, Nelson and Stephen in the context of randomly stirred fluids; in that case $\lambda = 1$.
- Applied to ASEP by van Beijeren, Kutner and Spohn (1985).

- Valid in the frame moving with the characteristics.
- In contrast to λ, ν is not a macroscopically defined coefficient; in principle we should let $\nu \to 0$ at the end of the calculation (in some sense).
- At this point it is not clear if the expansion of J to second order in u is sufficient (or necessary).

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b) The Kardar-Parisi-Zhang equation (1986)

The KPZ-equation is the interface analog of the stochastic Burgers eq. We derive it here from the hydrodynamic interface equation:

$$\frac{\partial h}{\partial t} = V\left(\frac{\partial h}{\partial x}\right), \quad V(u) = V(0) + \underbrace{V'(0)}_{c} u + \frac{1}{2} \underbrace{V''(0)}_{\lambda} u^2$$

$$\Rightarrow \frac{\partial h}{\partial t} = V(0) + c\,\frac{\partial h}{\partial x} + \frac{1}{2}\lambda\left(\frac{\partial h}{\partial x}\right)^2$$

Galilean transformation + vertical shift:

$$h(x,t) = V(0)\cdot t + \tilde{h}(x+ct,\, t)$$

$$\Rightarrow \frac{\partial \tilde{h}}{\partial t} = \frac{1}{2}\lambda\left(\frac{\partial \tilde{h}}{\partial x}\right)^2$$

Adding viscosity and noise and renaming $\tilde{h}$ by h we obtain the KPZ equation

$$\frac{\partial h}{\partial t} = \nu\,\frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \xi \qquad \text{KPZ}$$

$$\langle \xi(x,t)\,\xi(x',t')\rangle = D\,\delta(x-x')\,\delta(t-t')$$

Remark:

- Noise differs by a factor 2 from our definition of the Edwards-Wilkinson eq. in Sect. 6° c).
- KPZ was derived in 1986 assuming isotropic interface motion, $V(u) = V(0)\sqrt{1+u^2}$

$$\Rightarrow \lambda = V(0).$$

- First (time-dependent) exact solution was obtained by Sasamoto & Spohn in 2010.

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c) The Cole-Hopf transformation

Cole (1951) and Hopf (1950) discovered that the deterministic Burgers eq. can be mapped to a linear diffusion eq. through the transformation

$$Z(x,t) = \exp\left(\frac{\lambda}{2\nu} h(x,t)\right)$$

$$\Rightarrow \quad \frac{\partial Z}{\partial t} = \frac{\lambda}{2\nu}\left(\frac{\partial h}{\partial t}\right) Z$$

$$\frac{\partial Z}{\partial x} = \frac{\lambda}{2\nu}\left(\frac{\partial h}{\partial x}\right) Z$$

$$\frac{\partial^2 Z}{\partial x^2} = \left[\left(\frac{\lambda}{2\nu}\right)^2\left(\frac{\partial h}{\partial x}\right)^2 + \frac{\lambda}{2\nu}\left(\frac{\partial^2 h}{\partial x^2}\right)\right] Z =$$

$$= \frac{\lambda}{2\nu^2}\left[\frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \nu\frac{\partial^2 h}{\partial x^2}\right] Z$$

$$\Rightarrow \quad z^{-1}\frac{\partial z}{\partial t} = \left(\frac{\lambda}{2\nu}\right)\frac{\partial h}{\partial t} \overset{\text{KPZ}}{=}$$

$$= \left(\frac{\lambda}{2\nu}\right)\Big[\underbrace{\nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2}_{= \frac{2\nu^2}{\lambda}\frac{\partial^2 z}{\partial x^2} z^{-1}} + \xi\Big] =$$

$$= \nu \frac{\partial^2 z}{\partial x^2} z^{-1} + \frac{\lambda}{2\nu}\xi$$

$$\Rightarrow \quad \frac{\partial z}{\partial t} = \nu \frac{\partial^2 z}{\partial x^2} + \frac{\lambda}{2\nu}\xi(x,t)\cdot z \qquad (2)$$

linear diffusion eq. with multiplicative noise

The formal solution of (2) is a Wiener path integral (→ Exercise 20)

$$z(x,t) = \int_{(0,0)}^{(x,t)} \mathcal{D}X \, \exp\left[-S[X(t)]\right]$$

$$S[X(t)] = \int ds \left[\frac{1}{4\nu}\dot{X}(s)^2 - \frac{\lambda}{2\nu}\xi(X(s),s)\right]$$

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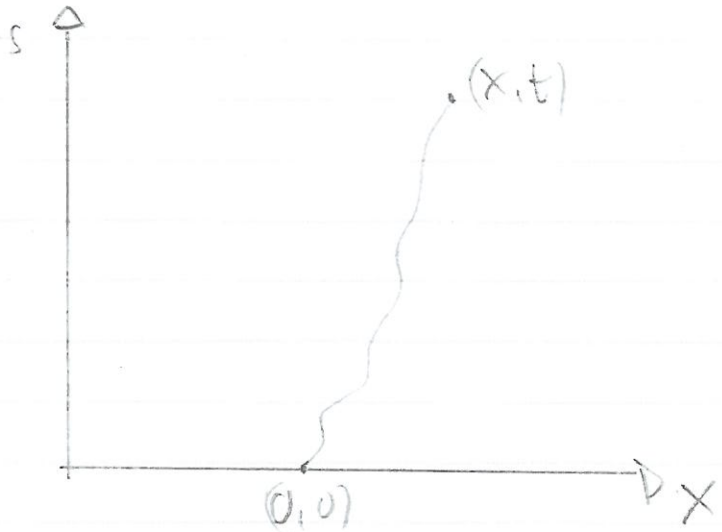
Interpretation: $X(s)$ is the configuration of a directed polymer with "line tension" $\frac{1}{\nu}$ in a random potential $\frac{\lambda}{2\nu}\, \xi(x,t)$



weight ("energy") of path is $S[X(s)]$

$\Rightarrow$ continuum analog of the waiting time representation in Sec. 5° b).

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d) The stationary distribution

Reminder: In the ASEP, the stationary distribution is the Bernoulli measure independent of the asymmetry q.

Here we show that a similar statement holds for the KPZ-equation.

To this end we need to consider the Fokker-Planck-eq. corresponding to the KPZ-eq.

Reminder: For a single variable Langevin eq.

$$\dot{X} = f(X) + \xi(t), \quad \langle \xi(t)\,\xi(t')\rangle = D\,\delta(t-t')$$

the equivalent Fokker-Planck-eq. reads

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} f(x)\, P(x,t) + \frac{1}{2} D \frac{\partial^2}{\partial x^2} P(x,t)$$

To generalize to stochastic PDE's, consider first the linear EW-eq. The Fourier coefficients evolve according to independent Ornstein-Uhlenbeck processes; as in Exercise 19, we consider a finite interface of length L with discrete Fourier modes:

$$\begin{cases} \dfrac{\partial}{\partial t} \hat{h}_n(t) = -\nu k_n^2\, \hat{h}_n(t) + \hat{\xi}_n(t), \quad n \in \mathbb{Z} \\[2ex] k_n = \dfrac{2\pi}{L} n, \quad \langle \hat{\xi}_n(t)\,\hat{\xi}_{n'}(t')\rangle = L D\, \delta_{n,-n'}\, \delta(t-t') \end{cases}$$

The FP-eq. for the probability distribution $P(\{\hat{h}_n\}, t)$ then reads

$$\frac{\partial}{\partial t} P(\{\hat{h}_n\},t) = -\sum_n \frac{\partial}{\partial \hat{h}_n}\left(-\nu k_n^2 \hat{h}_n P\right)$$

$$+ \frac{1}{2} DL \sum_n \frac{\partial^2}{\partial \hat{h}_n^2} P =$$

$$= \sum_n \frac{\partial}{\partial \hat{h}_n}\left\{\nu k_n^2 \hat{h}_n + \frac{DL}{2}\frac{\partial}{\partial \hat{h}_n}\right\} P .$$

The stationary distribution P^* satisfies (probability current $= 0$)

$$\left(\nu k_n^2 \hat{h}_n + \frac{DL}{2}\frac{\partial}{\partial \hat{h}_n}\right) P^* = 0$$

$\Rightarrow$ solution is a Gaussian:

$$P^*(\{\hat{h}_n\}) = \frac{1}{W} e^{-\frac{\nu}{DL}\sum_n k_n^2 |\hat{h}_n|^2}$$

In real space the exponent is

$$\frac{\nu}{DL}\sum_n k_n^2 |\hat{h}_n|^2 = \frac{\nu}{D}\int dx \left(\frac{dh}{dx}\right)^2$$

$$\Rightarrow \quad P^*(\{h(x)\}) = \frac{1}{W} e^{-\frac{\nu}{D}\int dx \left(\frac{dh}{dx}\right)^2}$$

$\Rightarrow$ Wiener measure with "diffusion constant" $D/2\nu$. This remains true for $L \to \infty$.

The KPZ nonlinearity adds a contribution to the RHS of the FP-eq., which reads

$$\sum_k \frac{\partial}{\partial \hat{h}_k} \left[-\frac{\hat{\lambda}}{2} \sum_q (k-q) q \, \hat{h}_{k-q} \, \hat{h}_q \right] P^*(\{\hat{h}_k\})$$

$$= -\frac{\hat{\lambda}}{2} \underbrace{\left(\sum_k \frac{\partial}{\partial \hat{h}_k} \sum_q (k-q) q \, \hat{h}_{k-q} \, \hat{h}_q \right)}_{=0,\ \text{because terms have either } q=0 \text{ or } q=k} P^*$$

$$- \frac{\hat{\lambda}}{2} \sum_{k,q} (k-q) q \, \hat{h}_{k-q} \, \hat{h}_q \frac{\partial}{\partial \hat{h}_k} P^*(\{\hat{h}_k\})$$

The second term is more easily analyzed in real space, where it reads

$$\frac{\lambda}{2} \int dx \left(\frac{\partial h}{\partial x} \right)^2 \underbrace{\frac{\delta}{\delta h(x)} \frac{1}{W} \exp\left[- \frac{\nu}{D} \int dx' \left(\frac{\partial h}{\partial x'} \right)^2 \right]}_{= \frac{2\nu}{D} \left(\frac{\partial^2 h}{\partial x^2} \right) \cdot P^*} =$$

$$= \left[\frac{\lambda \nu}{D} \underbrace{\int dx \left(\frac{\partial h}{\partial x} \right)^2 \left(\frac{\partial^2 h}{\partial x^2} \right)}_{= \frac{1}{3} \int dx \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial x} \right)^3 = 0} \right] \cdot P^* \qquad \square$$

Remark: As in the case of the ASEP, the proof relies crucially on periodic b.c.

We conclude that the stationary height difference correlation fct. of the KPZ-eq. is identical to the EW-result (Sect. 6° c)):

$$\langle (h(x,t) - h(x',t))^2 \rangle \xrightarrow[t\to\infty]{} \frac{D}{2\nu} |x-x'|$$

where the factor 4 comes from the difference in the definition of the noise.

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e) Statistical scale invariance

Reminder: The height difference correlation function of the EW-eq. has scaling form

$$G(x-x', t-t') = \langle (h(x,t) - h(x',t'))^2 \rangle =$$

$$= |x-x'| \, g(|t-t'|/|x-x'|^2) =$$

$$= |x-x'|^{2\alpha} \, g(|t-t'|/|x-x'|^z)$$

with $\alpha = \frac{1}{2}$, $z = 2$.

This reflects the invariance of the solution under rescaling of space, time and height in the following sense:

For a scale factor $b > 0$, define

$$\underline{\tilde{h}(x,t) = b^{-\alpha} h(bx, b^z t)}$$

then

$$\underline{\langle (\tilde{h}(x,t) - \tilde{h}(x',t'))^2 \rangle} = b^{-2\alpha} \langle (h(bx, b^z t) - h(bx', b^z t'))^2 \rangle$$

$$= b^{-2\alpha} |bx - bx'|^{2\alpha} g\left(b^z |t-t'| / b^z |x-x'|^z\right)$$

$$= \underline{\langle (h(x,t) - h(x',t'))^2 \rangle}$$

$\Rightarrow$ $h(x,t)$ and $\tilde{h}(x,t)$ have the same statistics for any b.

We will now argue that the KPZ-eq. has the same type of invariance but with <u>a different dynamic exponent z</u>. To this end we derive the stochastic PDE for $\tilde{h}$:

$$\left.\begin{aligned} \frac{\partial \tilde{h}}{\partial t} &= b^{z-\alpha} \frac{\partial h}{\partial t}, \quad \frac{\partial \tilde{h}}{\partial x} = b^{1-\alpha} \frac{\partial h}{\partial x}, \\ \frac{\partial^2 \tilde{h}}{\partial x^2} &= b^{2-\alpha} \frac{\partial^2 h}{\partial x^2} \end{aligned}\right\}$$

$$\Rightarrow \quad \frac{\partial \tilde{h}}{\partial t} = b^{z-\alpha} \Big[\nu \underbrace{\frac{\partial^2 h}{\partial x^2}}_{= b^{\alpha-2} \frac{\partial^2 \tilde{h}}{\partial x^2}} + \frac{\lambda}{2} \underbrace{\left(\frac{\partial h}{\partial x}\right)^2}_{= b^{2\alpha-2} \left(\frac{\partial \tilde{h}}{\partial x}\right)^2} + \zeta \Big] =$$

$$= b^{z-2}\nu \frac{\partial^2 \tilde{h}}{\partial x^2} + b^{z+\alpha-2}\frac{\lambda}{2}\left(\frac{\partial \tilde{h}}{\partial x}\right)^2 +$$

$$+ \underbrace{b^{z-\alpha}\,\xi(bx, b^z t)}_{=\tilde{\xi}(x,t)}$$

The correlation function of $\tilde{\xi}$ is

$$\langle \tilde{\xi}(x,t)\,\tilde{\xi}(x',t')\rangle = b^{2(z-\alpha)}\langle \xi(bx, b^z t)\,\xi(bx', b^z t')\rangle =$$

$$= b^{2(z-\alpha)} D\,\delta(b(x-x'))\,\delta(b^z(t-t')) =$$

$$= \underbrace{b^{2(z-\alpha)-1-z}}_{=b^{z-1-2\alpha}} D\,\delta(x-x')\,\delta(t-t').$$

We conclude that $\tilde{h}$ satisfies a KPZ-eq. with

renormalized coefficients

$$\tilde{\nu} = b^{z-2}\nu, \quad \tilde{\lambda} = b^{z+\alpha-2}\lambda,$$
$$\tilde{D} = b^{z-1-2\alpha} D$$

We now try to determine α and z by

demanding invariance of the coefficients:

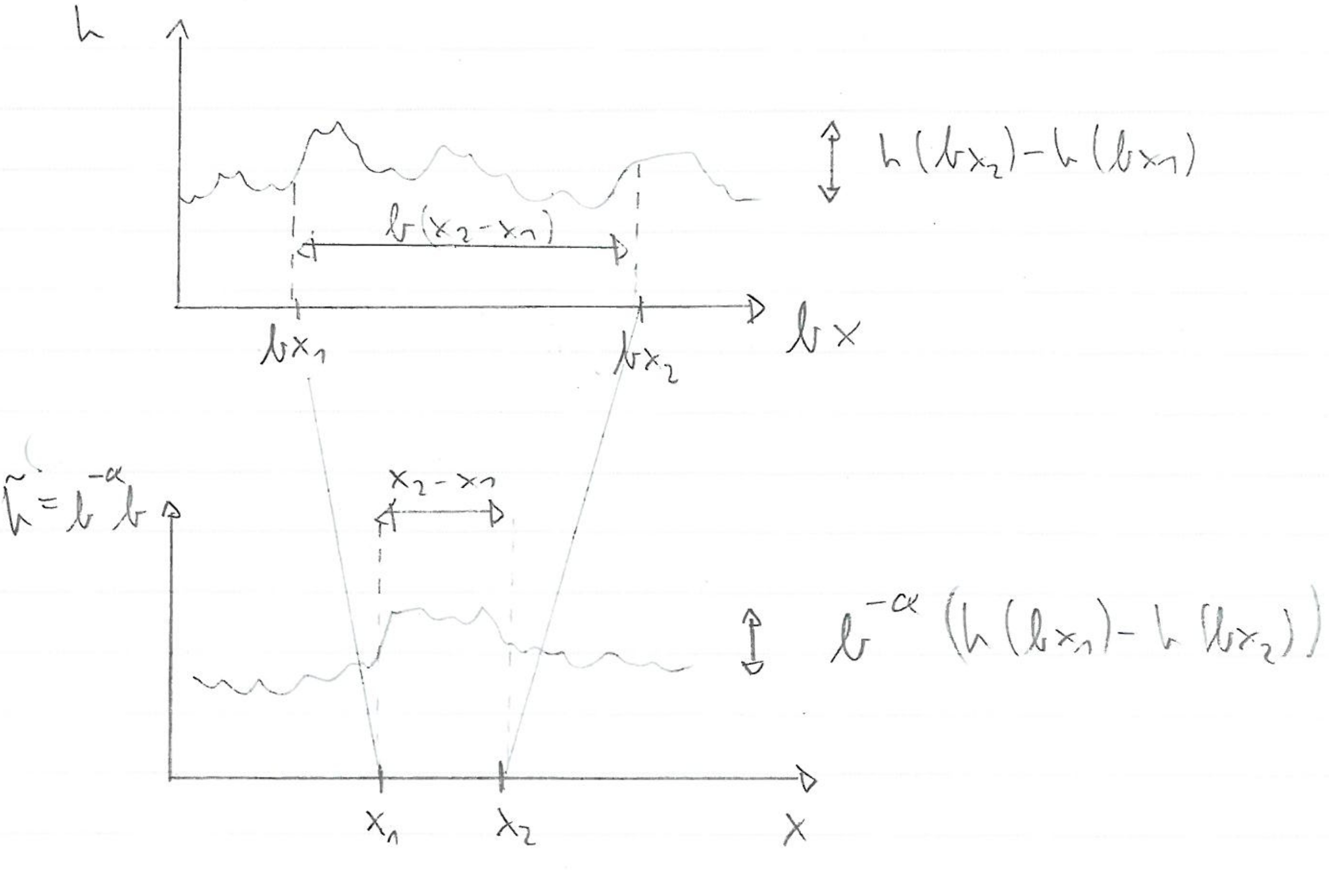
(i) $\lambda = 0$ (EW-eq.):

$$\left.\begin{array}{l} \tilde{\nu} = \nu \Rightarrow z = 2 \\ \tilde{D} = D \Rightarrow \alpha = \frac{1}{2}(z-1) = \frac{1}{2} \end{array}\right\} \checkmark$$

(ii) $\lambda \neq 0$: With $z = 2$, $\alpha = \frac{1}{2}$ we get

$\tilde{\lambda} = b^{1/2} \lambda \Rightarrow$ equation is not invariant.

What does this mean? Consider $b > 1$:



$\Rightarrow$ for $b > 1$ the transformation from h to $\tilde{h}$

is (the first step of) a coarse graining operation which extracts the behavior of the system on large length scales. The fact that λ grows under this operation implies that the term $\frac{\lambda}{2}\left(\frac{dh}{dx}\right)^2$ becomes more important on large scales, it is relevant in the renormalization group (RG) sense.

How to fix the exponents α and z for $\lambda \neq 0$?

We know that there are two invariants which should not change under rescaling:

- $\lambda = V''(0)$ is a macroscopically defined quantity which must be invariant:

$$\tilde{\lambda} = b^{z+\alpha-2}\lambda \overset{!}{=} \lambda \quad \Rightarrow \quad \alpha + z = 2$$

- The stationary distribution is governed by the ratio D/ν which therefore also must be invariant: $\frac{\tilde{D}}{\tilde{\nu}} = b^{2\alpha-1}\frac{D}{\nu} \quad \Rightarrow \quad \alpha = \frac{1}{2}$

$\Rightarrow$ the dynamic exponent is $z = \frac{3}{2}$.

Consequences:

- The height difference correlation function of the KPZ eq. is expected to have the scaling form

$$G(x-x', t-t') = |x-x'| \cdot g_{KPZ}\left(\frac{|t-t'|}{|x-x'|^{3/2}}\right)$$

The diffusive scaling of the EW-eq., $x \sim t^{1/2}$, is replaced by superdiffusive scaling $x \sim t^{2/3}$.

- In particular, the variance of the height fluctuations grows as

$$\langle (h(x,t) - h(x,t'))^2 \rangle \sim |t-t'|^{2\beta} = |t-t'|^{2/3}$$

with $\beta = \alpha/z = 1/3$ (cf. Sect. 6°c)).

- In the Cole-Hopf picture this implies superdiffusive wandering of the Brownian paths / directed polymers (cf. Sect. 7°c))