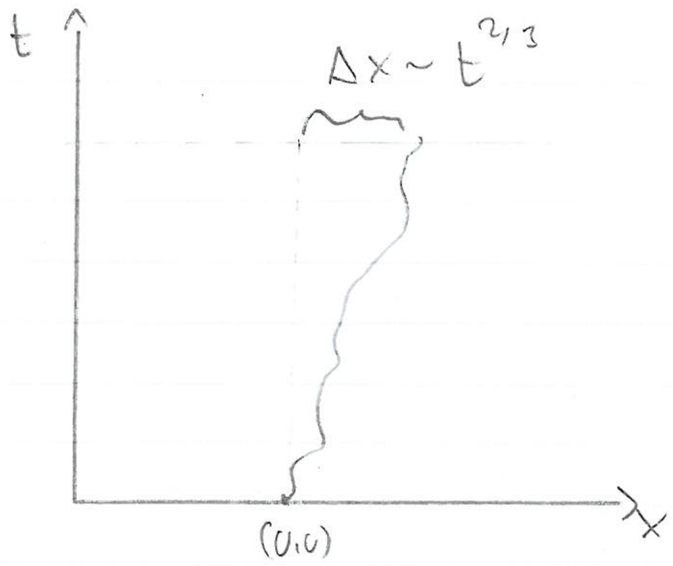

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f) Higher dimensions

KPZ-eq. for a d-dim interface (d-dim substrate plane) reads ($h = h(\vec{x}, t)$)

$$\left.\begin{aligned} \frac{\partial h}{\partial t} &= \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \xi \\ \langle \xi(\vec{x}, t)\, \xi(\vec{x}', t') \rangle &= D\, \delta^d(\vec{x} - \vec{x}')\, \delta(t - t') \\ \vec{x} &= (x_1, \ldots, x_d) \end{aligned}\right\}$$

We repeat the rescaling $h \rightsquigarrow \tilde{h}$ in this case.

Dimensionality only affects the noise term:

$$\tilde{\xi}(\vec{x}, t) = b^{z-\alpha}\, \xi(b\vec{x}, b^z t)$$

$$\Rightarrow \langle \tilde{\xi}(\vec{x}, t)\, \tilde{\xi}(\vec{x}', t') \rangle = b^{2(z-\alpha)}\, D\, \delta^d(b(\vec{x} - \vec{x}')) \times$$

$$\times\, \delta(b^z(t - t')) =$$

$$= \underbrace{b^{2(z-\alpha)}\, b^{-z-d}\, D}_{=\tilde{D}}\, \delta(\vec{x}-\vec{x}')\, \delta(t-t')$$

$$\Rightarrow \quad \underline{\tilde{D} = b^{z-d-2\alpha}\, D}$$

(i) $\underline{\lambda = 0 \text{ (EW-eq.)}}$: $\tilde{\nu} = \nu \Rightarrow z = 2$

$$\tilde{D} = D \Rightarrow \alpha = \frac{z-d}{2} = \underline{\frac{2-d}{2}}$$

α changes sign at $\underline{d=2}$, which implies a qualitative change of behavior: (see Exercise 22)

- $\underline{d < 2}$ $\Rightarrow$ $\langle h(x,t)\, h(x',t')\rangle$ infinite,
  $\langle (h(x,t) - h(x',t'))^2\rangle \sim |x-x'|^{2\alpha}$
- $\underline{d > 2}$: $\Rightarrow$ $\langle h(x,t)\, h(x',t')\rangle$ finite
  (in the presence of a microscopic cutoff),
  $\langle h(x,t)\, h(x',t')\rangle \sim |x-x'|^{2\alpha} = |x-x'|^{-(d-2)}$
  in the stationary state $t, t' \to \infty$
- $\underline{d = 2}$: Marginal case, logarithmic divergence of the height difference correlation fct.

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(ii) $\lambda \neq 0$: Inserting $z = 2$, $\alpha = \frac{2-d}{2}$ we have

$$\tilde{\lambda} = b^{z+\alpha-2} \lambda = b^{\alpha} \lambda = b^{\frac{2-d}{2}} \lambda$$

which suggests that the nonlinear term shrinks under rescaling for $d > 2$ and thus should be irrelevant on large scales.

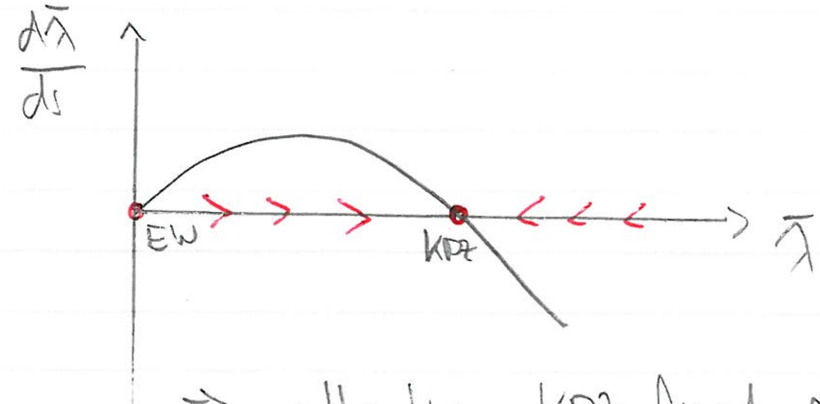
A perturbative RG treatment shows that this is not the case; (Forster, Nelson, Stephen 1977)

Flow eq. for the effective coupling $\bar{\lambda} = \lambda \sqrt{\frac{D}{2\nu^3}}$ under infinitesimal rescaling ($b = e^s$) reads

$$\frac{d\bar{\lambda}}{ds} = \frac{2-d}{2} \bar{\lambda} + A_d \frac{2d-3}{4d} \bar{\lambda}^3 \quad , A_d > 0$$

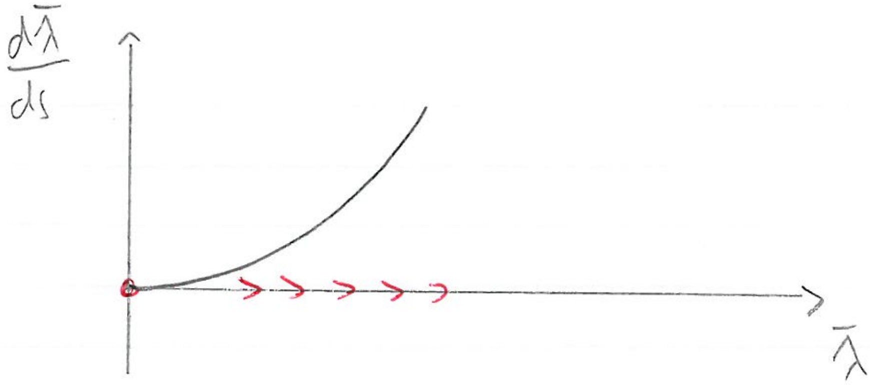
Phase portraits:

$d = 1$:



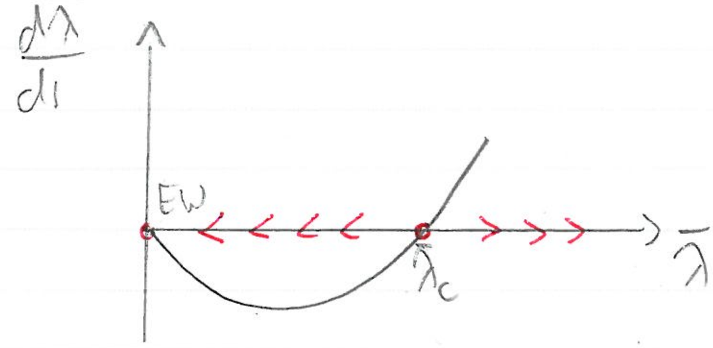
$\Rightarrow$ attractive KPZ fixed point with $\alpha = \frac{1}{2}$, $z = \frac{3}{2}$.

$d=2$:



$\Rightarrow$ EW-FP $\bar{\lambda}=0$ is marginally unstable, $\bar{\lambda}$ flows to a (unknown) KPZ-FP with exponents α_2, z_2, $\alpha_2+z_2=2$.

$d>2$:



$\Rightarrow$ EW-FP is stable for small λ, but beyond the critical coupling $\bar{\lambda}_c$ the system flows to a strong coupling phase with exponents α_d, z_d, $\alpha_d+z_d=2$.

Numerical estimates:

d	α_d
2	0.393 [1]
3	0.3135 [2]
4	0.255 [2]
5	0.205 [3]

[1] Kelling, Odor 2011
[2] Marinari et al. 2000
([3] Marinari et al. 2002)

Remark: The rescaling analysis can also be used to examine the importance of higher order terms, which were omitted in the expansion of the growth velocity $V(\nabla h)$. The coefficient λ_n of a term $\lambda_n |\nabla h|^n$ behaves as

$$\lambda_n \longrightarrow \tilde{\lambda}_n = b^{z-\alpha} \, b^{n(\alpha-1)} \lambda_n =$$
$$= \underline{b^{z-n+(n-1)\alpha} \lambda_n}$$

Inserting the EW exponents $z = 2$, $\alpha = \frac{2-d}{2}$ this gives $\tilde{\lambda}_n = b^{1-\frac{d}{2}(n-1)} \lambda_n$

$\Rightarrow |\nabla h|^3$ - term is marginal in $d=1$ and irrelevant in $d>1$.

At the KPZ - fixed point we have $\alpha + z = 2$ and therefore

$$\tilde{\lambda}_n = b^{(n-2)(1-z)} \lambda_n$$

which flows to zero when $n > 2$, $z > 1$.

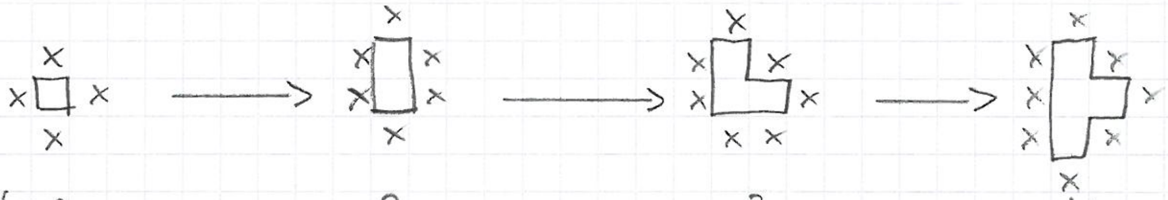
8° KPZ universality in 1+1 dimension

a) Growth models (1+1 dimensions)

(i) Eden model (M. Eden, 1961)

- Introduced to describe the growth of cell colonies
- Growth rule ($N \to N+1$): (on lattice)
  - identify all vacant neighboring sites of the current cluster (= perimeter sites)
  - fill one of the perimeter sites at random

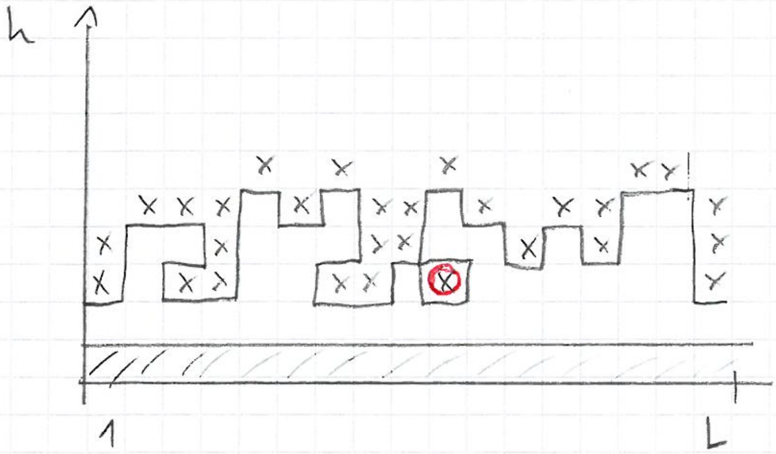
Growth from a seed: (cluster geometry)



$N=1$ 2 3 4 etc

x = perimeter site

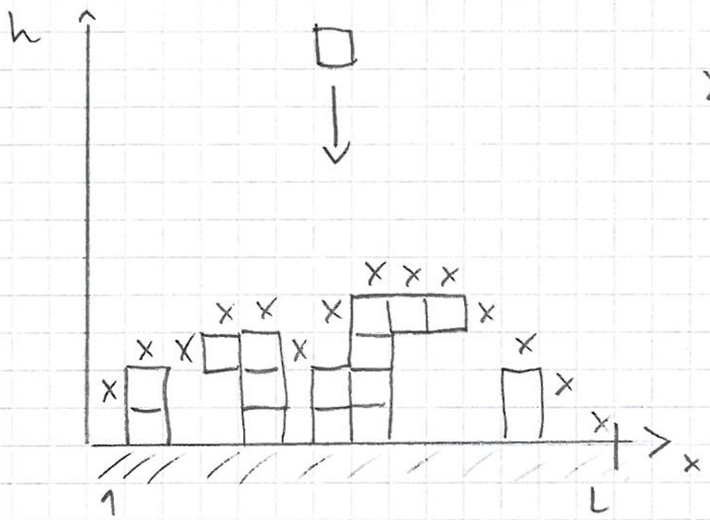
Growth from a line: (strip geometry)



⊗ = internal growth site

(ii) <u>Ballistic deposition</u> (M. Vold, 1959)

- Particles fall vertically downward along lattice columns and stick at the point of first contact with the existing deposit:



x = potential attachment sites for the next particle

<u>strip geometry</u>

In the <u>cluster geometry</u> all particles have to be connected to the first deposited particle

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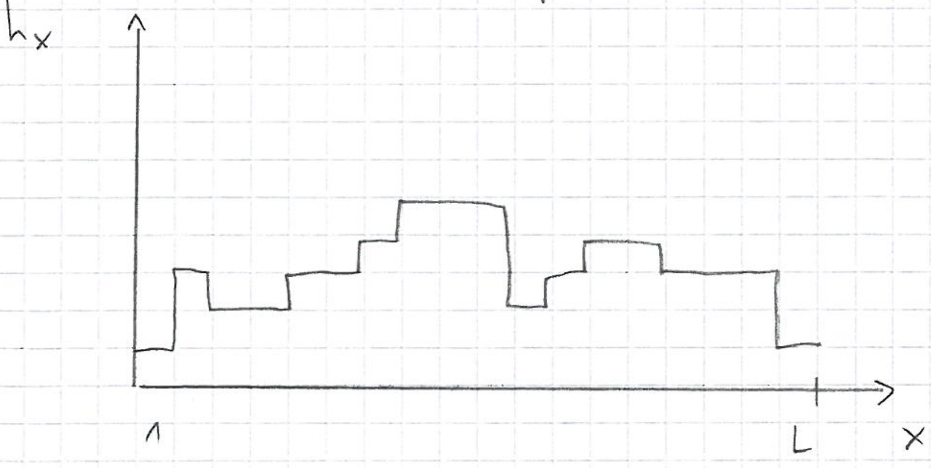
(iii) <u>Solid-on-solid (SOS) models</u>

- In SOS-models the surface is described by a single valued discrete height function without holes or overhangs.



- The simplest SOS-model is random deposition, where particles fall vertically downward along lattice columns and deposit above the previous particle in the same column (trivial Tetris)

$\Rightarrow$ $h_i(t)$ are uncorrelated Poisson processes.

- Restricted SOS (RSOS) model: (Kim, Kosterlitz '89)

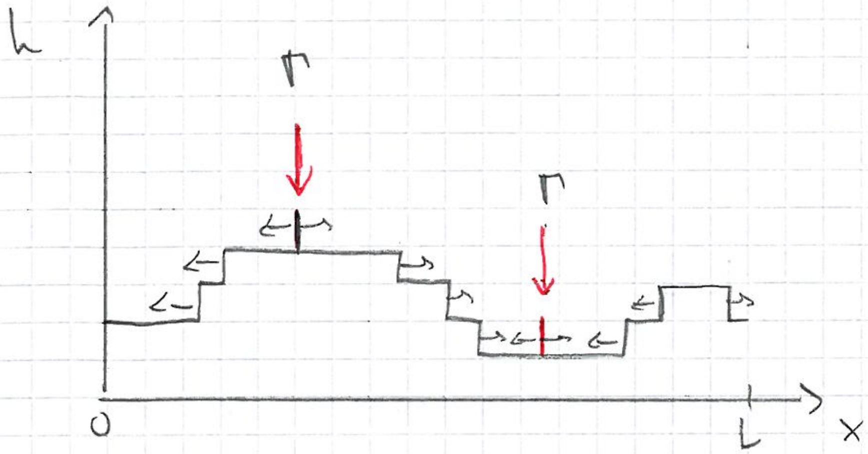
Deposition moves are accepted only if the RSOS condition

$$|h_{i+1} - h_i| \leq \Delta \quad , \quad \Delta = 1, 2, 3 \ldots$$

remains fulfilled at all times. The single step model introduced previously is a variant of the RSOS-model.

(iv) Polynuclear growth (PNG) model (Frank 1974)

- The PNG-model is discrete in height but continuous in space.
- "Islands" of height a and width 0 nucleate on the substrate or on other islands at rate Γ per unit time and length
- Once created, island edges spread laterally at speed c.
- Colliding islands coalesce.



strip geometry

In the droplet geometry only the first nucleation event occurs on the substrate, all later events occur on top of previously nucleated layors.

---

b) The universality hypothesis

KPZ conjecture: (1986)

- Any stochastic growth model with local rules and a nonlinear dependence of the growth velocity $V(u)$ as a fct. of inclination u is described on large length and time scales by the KPZ eq.

- In 1+1 dimensions this implies statistical scale invariance w.r.t. the transformation

$$h(x,t) \longrightarrow \tilde{h}(x,t) = b^{-\frac{1}{2}} h(bx, b^{3/2} t)$$

for any $b>0$, with the resulting constraints on the form of correlation functions.

Examples:

- Height difference correlation fct.: (infinite system)

$$G(r,s) = \langle (h(x,t) - h(x+r, t+s))^2 \rangle = |r|\, g(|r|/|s|^{2/3})$$

- Prob. density $f(h,t)$ of the height $h(x,t)$ (infinite system, flat initial condition):

$$f(h,t) = t^{-1/3} f((h-\langle h\rangle)/t^{1/3})$$

because $\langle (h-\langle h\rangle)^2\rangle \sim t^{2/3}$.

Refined universality conjecture: (~ 1992)

- In appropriate, model-dependent units for h, x and t, not only the scaling exponent $\alpha = \frac{1}{2}$ and $z = \frac{3}{2}$ are universal, but the entire random configuration $h(x,t)$ is a universal, spatio-temporal stochastic process.
- This implies in particular universality of scaling function like g and f defined above.

The appropriate model-dependent scale factors can be identified using dimensional analysis:

(i) Scale factors can only depend on the invariant quantities

$$\lambda = V''(u) \quad , \quad A := \frac{D}{2\nu}$$

(ii) The dimensions of λ and A can be read off from the KPZ-eq. (note that h and x have different dimensions):

$$\left.\begin{aligned} &\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \xi , \\ &\langle \xi(x,t)\, \xi(x',t') \rangle = D\, \delta(x-x')\, \delta(t-t') \end{aligned}\right\}$$

$$\left.\begin{aligned} \Rightarrow \quad & [\nu] = \frac{[x]^2}{[t]} , \quad [\lambda] = \frac{[x]^2}{[t][h]} \\ & [\xi] = \frac{[h]}{[t]} , \quad [D] = [x][t][\xi]^2 = \frac{[x][h]^2}{[t]} \end{aligned}\right\}$$

$$\Rightarrow [A] = \frac{[x][h]^2}{[t]} \cdot \frac{[t]}{[x]^2} = \frac{[h]^2}{[x]}$$

(iii) To find the scale factor for $f(L,t)$, we need a combination of A, λ and t that has the dimension of h:

$$\left.\begin{aligned} & [A^2 \lambda] = \frac{[h]^4}{\cancel{[x]^2}} \frac{\cancel{[x]^2}}{[t][h]} = \frac{[h]^3}{[t]} \\ & \Rightarrow \quad h \sim (A^2 \lambda t)^{1/3} \end{aligned}\right\}$$

$$\Rightarrow f(h,t) = (A^2|\lambda|t)^{-1/3} f_{KPZ}\left((h - \langle h \rangle) / A^2|\lambda| t^{1/3}\right)$$

with a universal (model-independent) function f_{KPZ}.

Exercise: Find the corresponding scale factors for $G(r,s)$

(iv) The coefficients A and λ can be computed analytically if the stationary distribution of the model is known, but in general they have to be determined numerically by measuring

- $V(u)$ from simulations of tilted substrates
- A from simulations of the stationary height difference correlation fct.

$$\lim_{t\to\infty} \langle (h(x,t) - h(x+r,t))^2 \rangle = A|r|$$

or the stationary surface width for a finite system of length L

$$\lim_{t\to\infty} \langle (h - \langle h \rangle)^2 \rangle = \frac{AL}{12}$$

(see Exercise 13).

Developments since 2000:

Exact solutions for

- dTASEP in corner growth geometry (K. Johansson, 2000)
- PNG-model (Prähofer & Spohn, 2000)

provide rigorous support for the refined universality conjecture, with the following qualifications:

(i) The universality class is geometry dependent.

The three main growth geometries are

I. Flat surface without fluctuations $(h(x,0) \equiv 0)$

II. Flat surface with stationary roughness (e.g., Bernoulli measure for ASEP)

III. Curved surface grown from a point seed (e.g., TASEP in corner growth geometry, or PNG droplet)

which lead to different scaling functions g_{KPZ}, f_{KPZ}.

(ii) The universal height distributions f_{KPZ} are related to the Tracy-Widom (TW) distributions discovered in 1994 in the context of extremal eigenvalues of random matrices.