c) Random matrices (based on Majumdar 2007)

- Random matrices were introduced by Wigner (1951) to describe complex spectra of atomic nuclei.
- Standard setting:

$N \times N$ matrices with Gaussian random entries and specified symmetry; most important examples are

GOE = Gaussian orthogonal ensemble of real symmetric matrices

GUE = Gaussian unitary ensemble of complex Hermitian matrices

- Joint distribution of eigenvalues: (Wigner 1951)

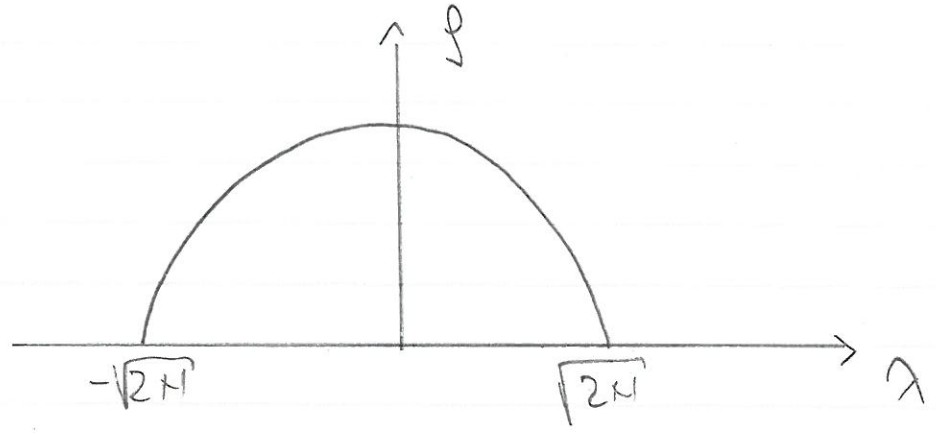
$$\begin{cases} P(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2} \left(\sum_{i=1}^{N} \lambda_i^2 - \sum_{i \neq j} \ln |\lambda_i - \lambda_j| \right) \right] \\ \text{GOE}: \ \beta = 1 \qquad \text{GUE}: \ \beta = 2 \end{cases}$$

What is the shape of the spectrum for large N?

- Average density of states:

$$\rho(\lambda, N) \xrightarrow[N\to\infty]{} \sqrt{\frac{2}{N\pi^2}} \left(1 - \frac{\lambda^2}{2N}\right)^{1/2}$$

Wigner semicircle



- Tracy & Widom (1994) consider the statistics of the largest (or smallest) eigenvalue, and show that

$$\lambda_{max} = \sqrt{2N} + \frac{1}{\sqrt{2}} N^{-1/6} \chi$$

where χ is a random variable with distribution

$$\left.\begin{array}{l} \mathrm{Prob}[\chi \leq x] = F_\beta(x) \quad , \quad \beta = 1,2 \\ F_\beta : \text{TW - distributions} \end{array}\right\}$$

In the KPZ context, F_1 describes class I, F_2 describes class III.

- <u>Shape of F_β:</u>

$$F_2(x) = \exp\left[-\int_x^\infty (y-x)\,u^2(y)\,dy\right] = e^{-g(x)}$$

u: Solution of the Painlevé II equation
$$u'' = 2u^3 + yu$$
with asymptotics $u(y) \sim \mathrm{Ai}(y)$, $y \to \infty$

$$F_1(x) = \exp\left[-\left(\tilde{g}(x) + g(x)\right)/2\right]$$
$$\tilde{g}'(x) = -u(x), \quad \tilde{g}(x) \to 0 \text{ for } x \to \infty$$

The densities $f_\beta = \frac{d}{dx} F_\beta$ have <u>non zero mean and asymmetric, non-Gaussian tails:</u>

$$\ln f_\beta(x) \sim \begin{cases} -c_\beta |x|^3 & x \to -\infty \\ -d_\beta |x|^{3/2} & x \to \infty \end{cases}$$

with $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{12}$

$$d_1 = d_2 = \frac{4}{3}$$

- Characterization by cumulants:

$$\left.\begin{aligned} &\langle X\rangle_c = \langle x\rangle, \quad \langle X^2\rangle_c = \langle (x-\langle x\rangle)^2\rangle \\ &\langle X^3\rangle_c = \langle (x-\langle X\rangle)^3\rangle, \\ &\langle x^4\rangle_c = \langle (x-\langle x\rangle)^4\rangle - 3\langle X^2\rangle_c \end{aligned}\right\}$$

Deviations from the Gaussian are quantified by

$$\left.\begin{aligned} &\text{skewness} \quad s = \frac{\langle X^3\rangle_c}{\langle X^2\rangle_c^{3/2}} \\ &\text{kurtosis} \quad \kappa = \frac{\langle X^4\rangle_c}{\langle X^2\rangle_c^2} \end{aligned}\right\} \quad \text{for a Gaussian } s = \kappa = 0.$$

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d) The Ulam problem

Consider a permutation π of $\{1,..,N\}$:

$$\pi : \{1,..,N\} \longrightarrow \{\pi(1),..,\pi(N)\}$$

What is the longest increasing subsequence (LISS) of π, i.e. the largest subset $\{i_1,..,i_\ell\} \subset \{1,..,N\}$ such that $\pi(i_1) < \pi(i_2) < \ldots < \pi(i_\ell)$?

Example ($N=10$):

$\{\pi(1), \ldots, \pi(10)\} = \{7, 1, 3, 8, 5, 4, 2, 10, 6, 9\}$

Increasing subsequences: $\{1, 3, 5, 6, 9\} \Rightarrow l = 5$

$\{1, 3, 4, 10\}, \{1, 3, 4, 9\} \Rightarrow l = 4$

$\Rightarrow$ [illegible] + shorter subsequences

$\Rightarrow$ length of LISS is $\underline{l_N = 5}$.

An algorithm to determine l_N:

- decompose the permutation into disjoint decreasing subsequences by starting at the first entry and always choosing the nearest smaller entry to the right:

$\{7, 1, 3, 8, 5, 4, 2, 10, 6, 9\}$

$\downarrow$

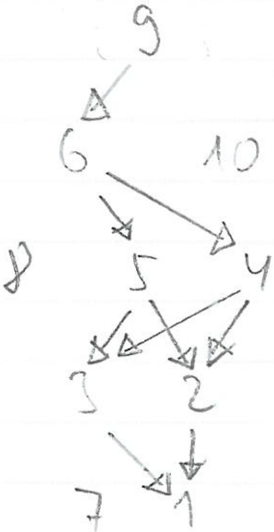
$\{7, 1\}, \{3, 2\}, \{8, 5, 4\}, \{10, 6\}, \{9\}$

- then the number of decreasing subsequences is equal to the length l_N of the LISS.

Proof: Denote by n_N the number of decreasing subsequences (DSS's).

(i) $n_N \geq l_N$: Entries of any ISS must lie in different DSS's.

(ii) $l_N \geq n_N$: It is always possible to construct one or several LISS's of length n_N going backwards along the DSS's:



$\square$

Ulam's problem: What is the statistics of l_N for random permutations, drawn with equal weight from all $N!$ elements of S_N?

[obviously $1 \leq l_N \leq N$]

History:

- S. Ulam (1961): $l_N \sim \sqrt{N}$ based on simulations
- Hammersley (1972): $l_N \to c\sqrt{N}$ with $\frac{\pi}{2} < c < e$
- Kingman (1973): $1.595 \le c \le 2.49$
- Logan & Shepp (1977); Vershik & Kerov (1977): $c = 2$
- Baik, Deift, Johansson (1999): For $N \to \infty$

$$l_N = 2\sqrt{N} + \chi \cdot N^{1/6} \quad \text{with } \mathbb{P}(\chi \le x) = F_2(x)$$

where F_2 is the GUE TW-distribution.

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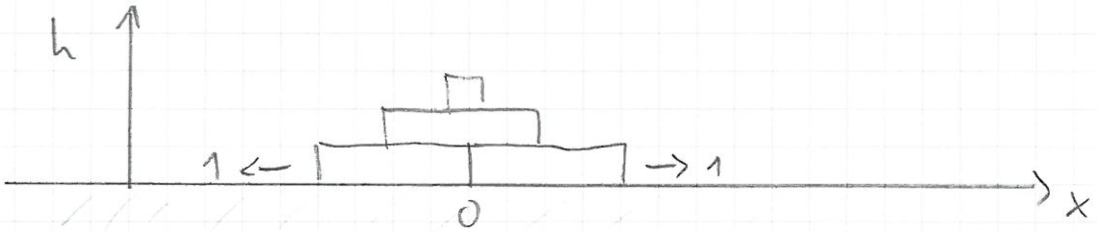
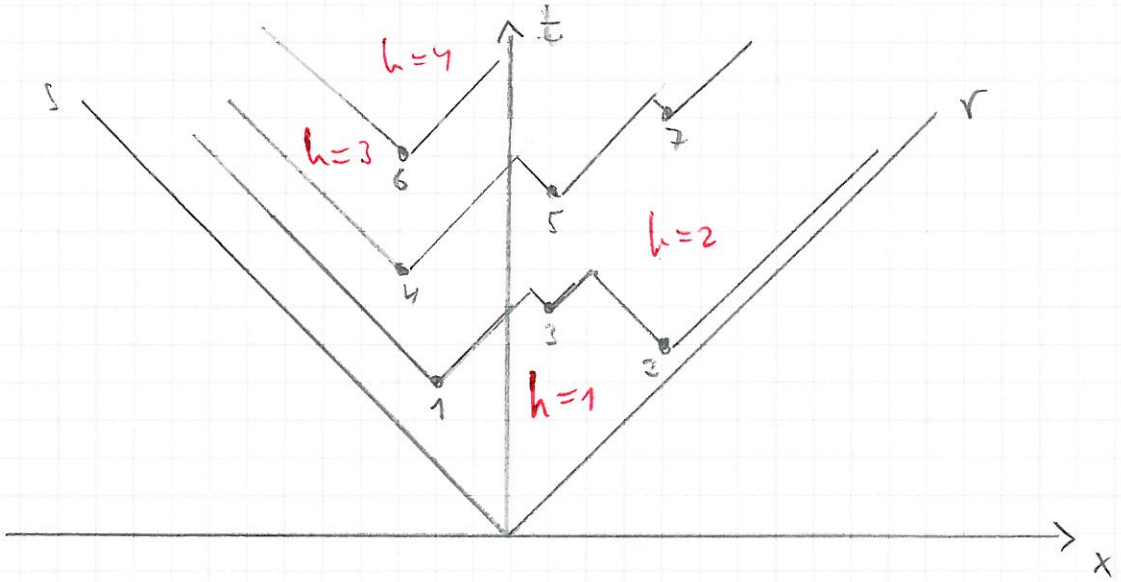
e) Random permutations and the PNG-model

- Prähofer & Spohn (2000) map the PNG-model to the Ulam problem, thus establishing that KPZ height fluctuations follow the TW distributions.
- We consider PNG-model in the droplet geometry and set the spreading speed $c = 1$ for convenience.



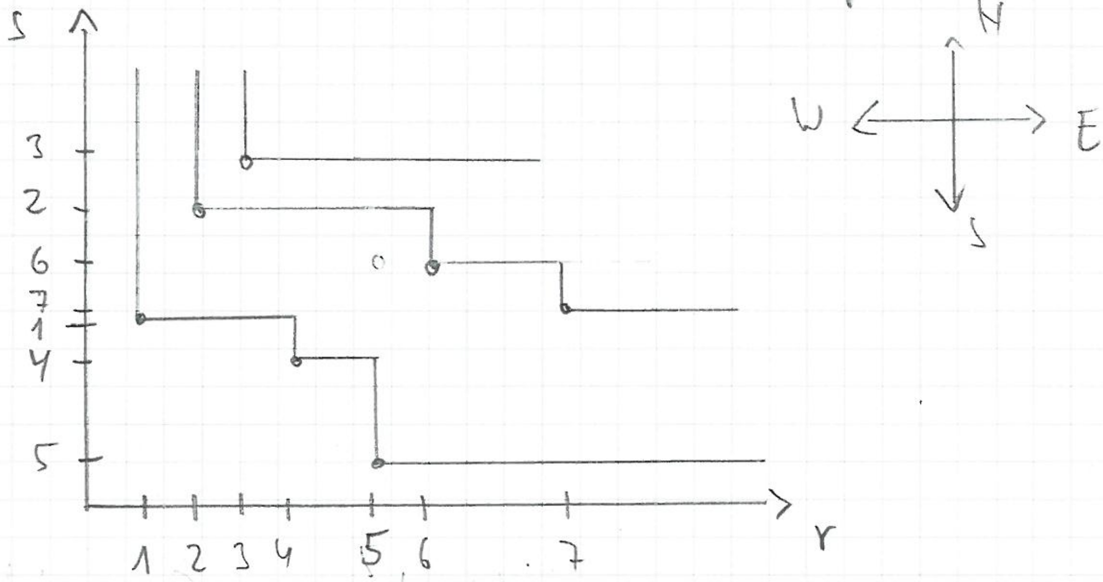
World lines of steps in the PNG-model:



- First nucleation event at $x=0$, $t=0$

- Subsequent events form a two-dimensional "Poisson process with density Γ in the "forward light cone" $-t \le x \le t$.

• To map to random permutations we turn the figure by 45° (cf. TASEP $\to$ corner growth):



- Labeling the nucleation events according to their order along the r-axis, the order along the s-axis induces a random permutation:

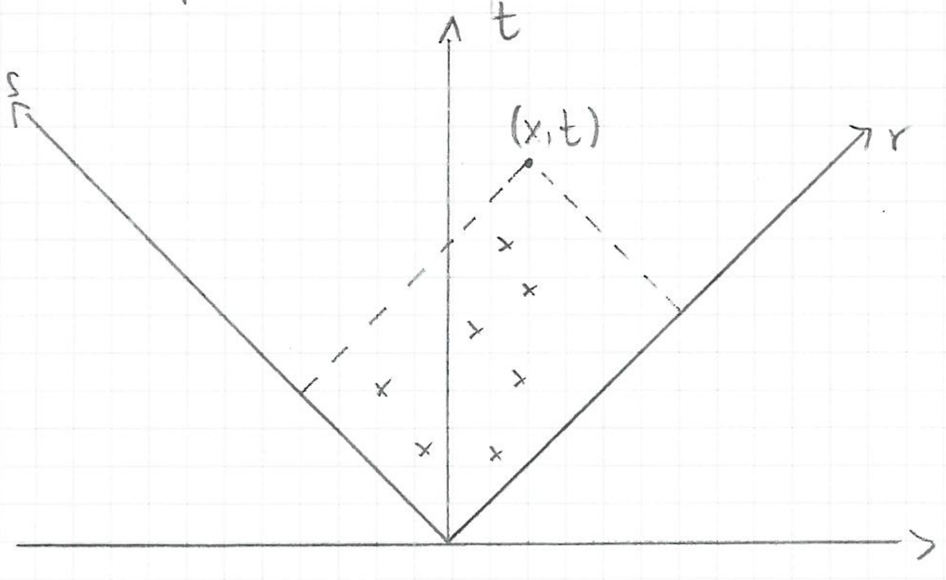
$$\pi: \{1,2,3,4,5,6,7\} \to \{5,4,1,7,6,2,3\}$$

- The world lines are SE-directed paths which connect the nucleation events within a DSS:

$$\{5,4,1,7,6,2,3\} \to \{5,4,1\}, \{7,6,2\}, \{3\}$$

$\Rightarrow$ number of world lines =
number of grown layers =
number of DDS's =
surface height $-1 = \ell_N$

- To be precise, return to the (x,t)-plane:



$$r = \frac{1}{\sqrt{2}}(t+x)$$

$$s = \frac{1}{\sqrt{2}}(t-x)$$

- $h(x,t)-1$ is the length of the LISS of the permutation induced by the nucleation events in the rectangle

$$\left\{ 0 \le r \le \frac{1}{\sqrt{2}}(t+x)\,,\ 0 \le s \le \frac{1}{\sqrt{2}}(t-x) \right\}$$

with area $\frac{1}{2}(t^2-x^2)$

- The number of nucleation events in this area is a Poisson-distributed RV with mean

$$\bar{N} = \frac{1}{2}\Gamma(t^2-x^2)$$

$$\Rightarrow \langle l_{\bar{N}} \rangle = \langle h(x,t) \rangle = \sqrt{2\Gamma(t^2-x^2)} = 2\sqrt{\bar{N}}$$

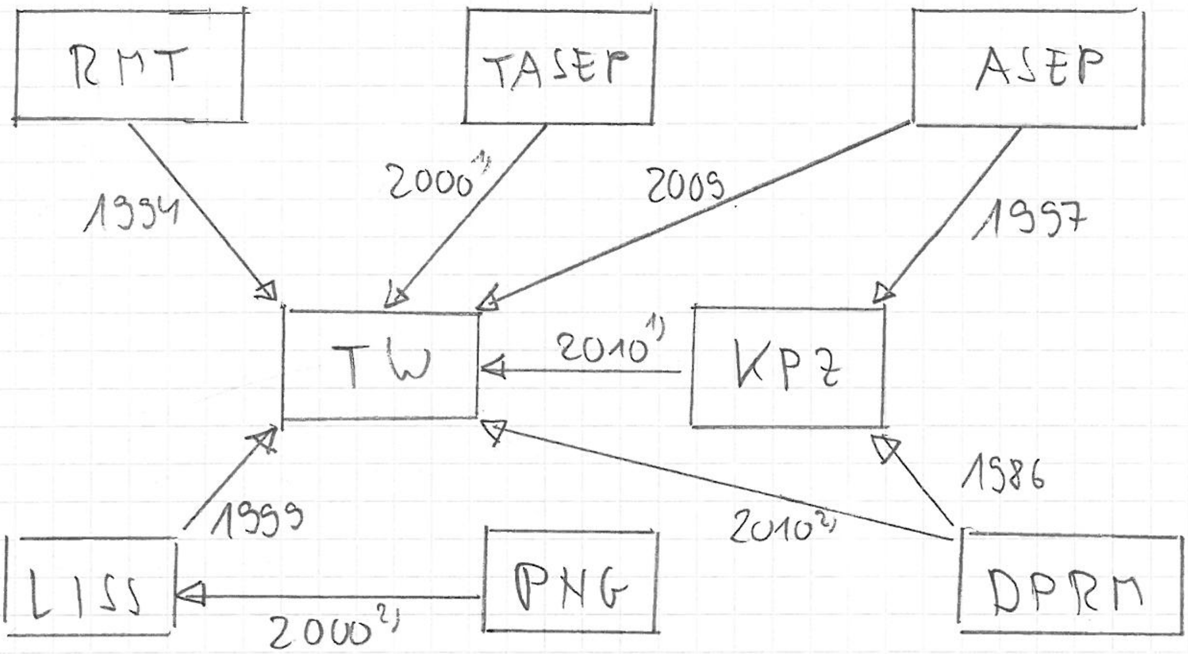
(PNG growth shape from Exercises 16 and 24)

which explains why $l_N \sim \sqrt{N}$ and proves that the prefactor is $c = 2$.

- Conversely, the theorem of Baik, Deift and Johansson proves that height fluctuations of the PNG droplet are distributed according to the TW-distribution F_2

- Geometry classes I and II require some additional work.

Summary



RMT: Random Matrix Theory

Key papers:

1994 Tracy & Widom
1997 Bertini & Giacomin
1999 Baik, Deift, Johansson
2000[1] Johansson
2000[2] Prähofer & Spohn
2009 Tracy & Widom
2010[1] Sasamoto & Spohn
2010[2] Calabrese, Le Doussal & Rosso