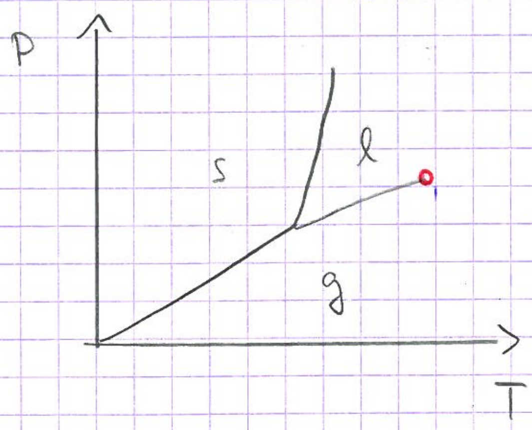
III. Scale invariance far from equilibrium

So far (< 1980): Scale invariance at critical points in thermal equilibrium



- $\xi = \infty$ ⇒ no characteristic length scale other than a, L
- Power law decay of correlations

$$G(r) \sim r^{-(d-2+\eta)}$$

- System is invariant under rescaling ⇒ RG fixed point

Scale invariance at equilibrium critical points requires fine tuning of parameters: $(P, T) = (P_c, T_c)$

⇒ non-generic behavior

Now (> 1980): Concepts & mechanisms for generic scale invariance in nonequilibrium systems

Three historical roots:

- Fractal geometry ⇔ scale invariance in space (B. B. Mandelbrot)
- Scale-invariant temporal processes ($1/f$ - noise)
- Power-law "event size" distributions (Pareto (1896); Zipf (1949); Gutenberg-Richter (1944))

Bilder: Newman, Bak

1° Scale invariance, self-similarity & power laws

Consider some "observable" F which depends on some parameter x, $F = F(x)$.

Examples:
- Probability distribution (F) of sizes (x) of earthquakes, wealth of individuals, ..
- Correlation between two spins at distance x

Invariance under rescaling of x implies

$$\circledast \qquad F(bx) = g(b)\, F(x) \qquad \text{for any } x,\ b>0$$

- Self-similarity: The visual impression of a structure reflects the relative number of features of different sizes:

$$\circledast \Rightarrow \frac{F(bx)}{F(x)} = g(b) \quad \text{independent of } x$$

$\Rightarrow$ structure "looks the same" on all scales

Bild: BD

- Power laws: If $\circledast$ is true for any $b>0$ we have continuous scale invariance.

Taking the derivative on both sides we have

$$x\, F'(bx) = g'(b)\, F(x)$$

$\underline{b = 1:}$ $\quad \dfrac{dF}{dx} = g'(1)\,\dfrac{F(x)}{x}$

$$\Rightarrow \ln F = g'(1) \ln x + \tilde{C}$$

$$\Rightarrow \underline{F(x) = C\, x^{\alpha}} \ , \quad \alpha = \frac{g'(1)}{g(1)} \ , \quad g(1) = 1$$

and $\quad g(b) = \dfrac{F(bx)}{F(x)} = b^{\alpha}$.

- Generalizations:

a) Discrete scale invariance (D. Sornette, 1998)

Suppose ⊛ is valid only for a specific value of b. Then rewrite ⊛ by setting

$$\phi = \ln F, \quad \xi = \ln x, \quad \beta = \ln b, \quad \gamma = \ln g$$

$$\Rightarrow \underline{\phi(\xi + \beta) = \gamma + \phi(\xi)}$$

$\Rightarrow$ discrete translational symmetry in $\ln x = \xi$

The general solution reads

$$\left.\begin{aligned} \phi(\xi) &= \frac{\gamma}{\beta}\,\xi + \psi(\xi) \\ \psi(\xi + \beta) &= \psi(\xi) \quad \text{periodic} \end{aligned}\right\}$$

$$\Rightarrow \underline{F(x) = x^{\gamma/\beta}\, P(\ln x)} \ , \quad P = e^{\psi} \ , \quad P(\xi) = P(\xi + \beta)$$

$\uparrow$ log-periodic function

Setting e.g. $P(\xi) = A\cos(2\pi\xi/\beta)$

$$\Rightarrow \quad F(x) = \frac{1}{2} A x^{\alpha} \left(x^{i\frac{2\pi}{\beta}} + x^{-i\frac{2\pi}{\beta}} \right)$$

$\Rightarrow$ "complex scaling exponent" $\alpha = \frac{\gamma}{\beta} \pm i\,\frac{2\pi}{\beta}$

b) Several variables

The generalization of * to n variables reads

$$F(b^{y_1} x_1, \ldots, b^{y_n} x_n) = g(b)\, F(x_1, \ldots, x_n)$$

The composition law $g(b_1 b_2) = g(b_1)\, g(b_2)$ fixes g:

$$g(b) = b^{\alpha}$$

Setting e.g. $b^{y_n} x_n = 1$ eliminates one variable:

$$F(x_1, \ldots, x_n) = b^{-\alpha} F(b^{y_1} x_1, \ldots, b^{y_{n-1}} x_{n-1}, 1) =$$

S

$$= x_n^{\alpha/y_n} \cdot F\left(x_1 / x_n^{y_1/y_n}, \ldots, x_{n-1} / x_n^{y_{n-1}/y_n}, 1\right)$$

but does not generally further constrain F.

Generalization to local scale invariance:

(Tang, 1989; Sitter & Hinrichsen, 2002)

$$F(b^{y_1(x_1,\ldots,x_n)} x_1, \ldots, b^{y_n(x_1,\ldots,x_n)} x_n) =$$

$$= b^{\alpha} F(x_1, \ldots, x_n)$$

Then the group property of the rescaling transformation implies that

$$(\vec{y}\cdot\nabla)\,\vec{y} = 0$$

in log. coordinates ! Problem

which allows also for logarithmic scaling forms

$$\frac{\ln F(x_1,\dots,x_n)}{\ln x_n} = G\left(\frac{\ln x_1}{\ln x_n},\dots,\frac{\ln x_{n-1}}{\ln x_n}\right)$$

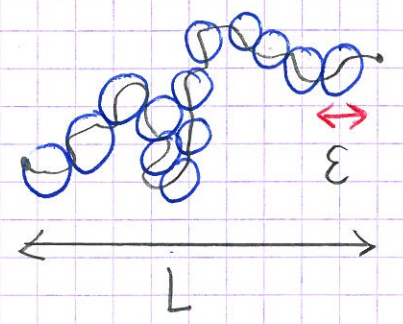
replacing the standard form S.

2° Fractal geometry

B. B. Mandelbrot (1967): "How long is the coast of Britain?"

Bilder: Bak, Mandelbrot + citation

The length / area / volume of irregular structures in nature depends on the resolution of the measurement:



$$l(\varepsilon) = \varepsilon\,N(\varepsilon) \sim \varepsilon^{1-D} \xrightarrow[\varepsilon\to 0]{} \infty$$

$$N(\varepsilon) \sim \left(L/\varepsilon\right)^D \quad \Big| \quad \text{if } D > 1$$

$N(\varepsilon)$: Number of disks required to cover the line

Generally: "Mass" $M(\varepsilon) = C\,\varepsilon^{d_t}\left(L/\varepsilon\right)^D$

$d_t = 0, 1, 2, 3, \ldots$: topological dimension
(here $d_t = 1$)

D : capacity dimension / Hausdorff dimension

Precise definition: Consider the quantity

$$\mathcal{M}(d) := \lim_{\varepsilon \to 0} \Big(\inf_{\{B_m : |B_m| \leq \varepsilon\}} \sum_m |B_m|^d \Big)$$

$\{B_m\}$: Set of balls covering the set

$|B_m|$: Radius of B_m

Then clearly

$$\mathcal{M}(d) = \lim_{\varepsilon \to 0} \varepsilon^{d-D} = \begin{cases} \infty & d < D \\ 0 & d > D \end{cases}$$

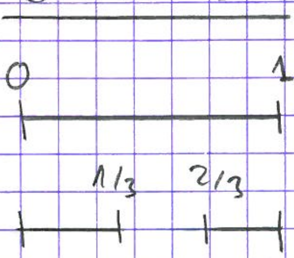
which defines D.

A fractal is a set whose Hausdorff dimension exceeds its topological dimension, $D > d_t$.

RW: $d_t = 1$, $D = 2$

Examples of deterministic fractals:

(i) Cantor set $(d_t = 0)$



$\Downarrow$

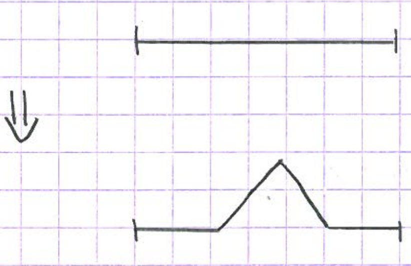
$\varepsilon = 1$, $N(\varepsilon) = 1$

$\varepsilon = 1/3$, $N(\varepsilon) = 2$

$$\Rightarrow \varepsilon_n = 3^{-n}, \; N_n = 2^n \quad \Rightarrow \quad D = \frac{\ln 2}{\ln 3} \simeq 0.63$$

Problem: Gap size distribution

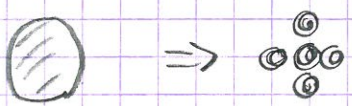
(ii) Koch curve ($d_t = 1$)



$$\left.\begin{array}{l} \varepsilon = 1, \quad N = 1 \\ \\ \varepsilon = \frac{1}{3}, \quad N = 4 \end{array}\right\}$$

$$\Rightarrow \quad D = \frac{\ln 4}{\ln 3} \simeq 1.26$$

(iii) Vicsek fractal Bild / Problem: d dimensions



$$\left.\begin{array}{l} \varepsilon_n = \frac{1}{3}\,\varepsilon_{n-1}, \quad N_n = 5\,N_{n-1} \\ \Rightarrow \quad D = \frac{\ln 5}{\ln 3} \simeq 1.465 \end{array}\right\}$$

- Deterministic fractals show discrete scale invariance.
- Natural fractals show continuous scale invariance in the statistical sense.

Example: Sedimentary layers (Sornette) Bild

Challenges for theory:

(i) Develop descriptive mathematical models for random fractal structures

(ii) Identify mechanisms for the formation of such structures (Kadanoff, 1986)

23.1.2008

A note of caution: Bild: Biber

Scale invariance in the mathematical sense is an asymptotic property ($\varepsilon \to 0$, $N \to \infty$) but in real structures it is limited to a finite scaling range

$$a \leq x \leq L \quad \text{with} \quad \frac{L}{a} \simeq 10 - 100$$

9.1.07

3° Diffusion-limited aggregation (DLA)

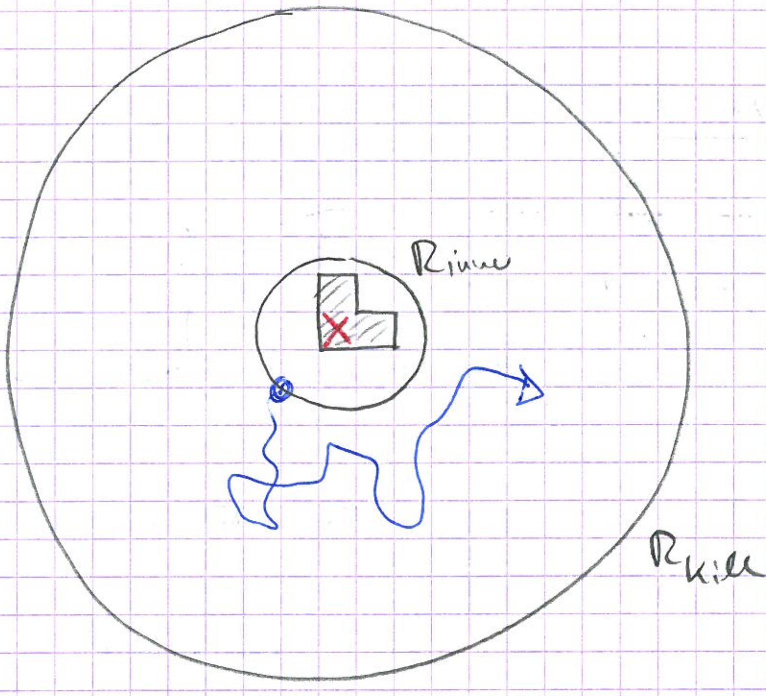
(T. A. Witten, L. M. Sander, 1981)

3.1. Basic phenomena

Algorithm:
(i) Place seed particle at origin
(ii) Start RW from a random point on a circle surrounding the current cluster
(iii) If RW hits the cluster it sticks irreversibly
(iv) If RW crosses the killing radius R_{kill} it is terminated

Exact implementation requires $R_{kill} \to \infty$, or use of the first return probability to the inner circle.

Bild: Regge



Results for the fractal dimension

$d=2$: $D \simeq 1.71$ off-lattice
$D \to 3/2$ square lattice

Bilder

$\Rightarrow$ lattice structure remains asymptotically relevant !

$d>2$: $d-1 < D < d$ for $d = 3-8$

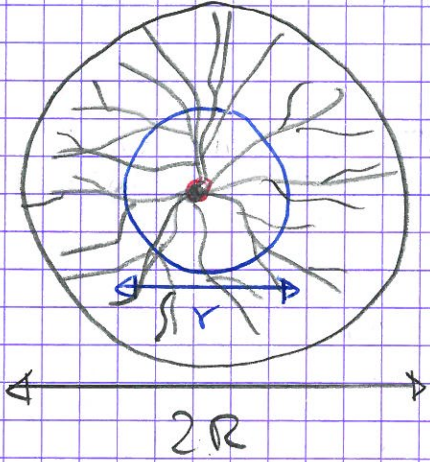
d	3	4	5	6
D	2.50	3.40	4.33	5.4

(Tolman and Meakin, 1989)

- $D < d$ because of screening: Fjords cannot fill up Bild: Mandelbrot
- $D > d-1$ $\Rightarrow$ no upper critical dimension $d_>$ such that $D =$ const. for $d > d_>$ (cf SAW)

3.2 Bounds on the fractal dimension

a) Opacity bound (Witten, Sander 1983)



Number of particles within distance r from the origin is $n(r) \sim r^D$

$\Rightarrow$ density profile

$$\rho(r) = \frac{n}{r^d} \sim r^{-(d-D)}$$

How far can a RW penetrate into the cluster?

Density at the boundary is $\rho(R) \sim R^{-(d-D)}$

$\Rightarrow$ RW is absorbed after $N_{RW} = 1/\rho \sim R^{d-D}$ steps

$\Rightarrow$ penetration length $\xi = \sqrt{N_{RW}} \sim R^{\frac{1}{2}(d-D)}$

Consistency requires $\xi < R \Rightarrow d - D < 2$

$\Rightarrow D > d-2$

For $D < d-2$ the cluster would become transparent to the RW.

Remark: If R were the only length scale of the structure (apart from the particle size) then $\xi \sim R \Rightarrow D = d-2$!

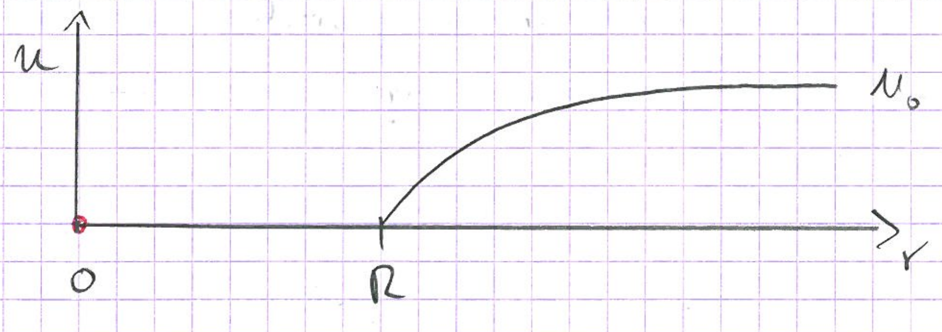
$\Rightarrow$ DLA is probably not strictly self-similar.

l) Causality bound (Ball & Witten, 1984)

Assume $D > d-2 \Rightarrow \xi \ll R$

$\Rightarrow$ cluster absorbs RW's like a sphere of radius R.

Assume a small constant density u_0 of walkers outside of the cluster:



In the stationary state $u(r)$ is the solution of

$$\frac{\partial u}{\partial t} = \underbrace{\nabla^2 u = 0}_{\text{Laplace eq.}} \Rightarrow \frac{\partial}{\partial r} r^{d-1} \frac{\partial u}{\partial r} = 0$$

$$\Rightarrow \quad u(r) = u_0 \left(1 - \left(\frac{R}{r}\right)^{d-2}\right) \qquad (d>2)$$

The flux onto the sphere per unit area is

$$\left.\frac{\partial u}{\partial r}\right|_{r=R} = \frac{u_0 (d-2)}{R}$$

$$\Rightarrow \text{ total flux } \quad \frac{dN}{dt} \sim R^{d-1} \left.\frac{\partial u}{\partial r}\right|_{r=R} \sim u_0 R^{d-2}$$

$$\Rightarrow \frac{dR}{dt} \sim \frac{d}{dt} N^{1/D} \sim R^{1-D} \frac{dN}{dt} \simeq u_0 R^{d-D-1}$$

$u_0 \to 0, \; R \to \infty$

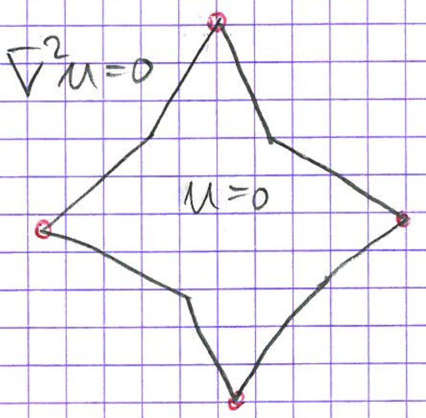
$\Rightarrow$ bounded growth speed requires

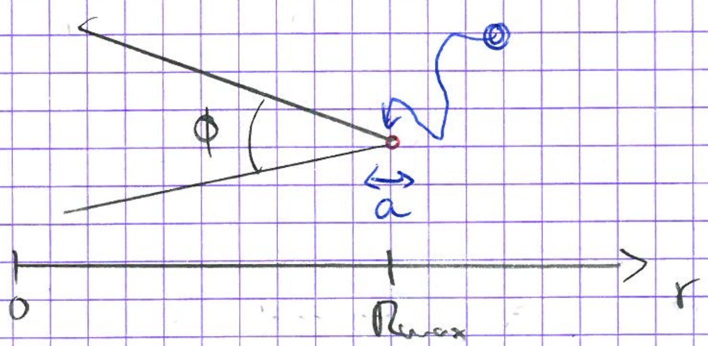
$$D > d-1$$

Rigorous proof by H. Kesten (1990).

c) Tip growth ($d=2$) (Turkevich & Scher, 1985)

Consider the growth of the maximal cluster radius R_{max}. This increases only if a particle attaches within distance a from the tip:



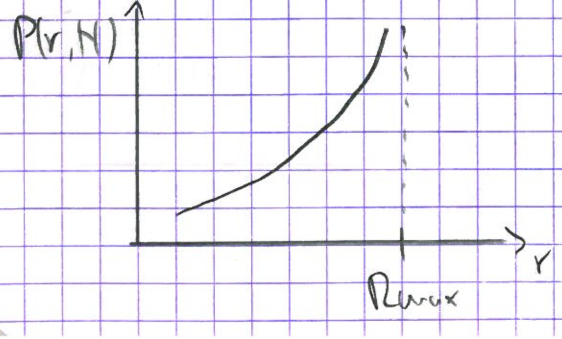


The tip is modeled as a wedge of opening angle ϕ

$P(r, N)$: Prob. that the next walker attaches at distance r from the center

$P(r,N) \sim \vec{n} \cdot \nabla u|_r \sim$ normal electric field associated with electrostatic potential u

Electrostatics $\Rightarrow$ $P(r, N) \sim (R_{max} - r)^{-\nu}$



$$\nu = \frac{\pi - \phi}{2\pi - \phi}$$

Normalization: $\int_0^{R_{max}} dr\, P(r,H) = 1$

$$\Rightarrow \quad P(r,H) \sim R_{max}^{-(1-\nu)} \left(R_{max} - r\right)^{-\nu}$$

Growth probability at the tip:

$$P_{tip} \simeq \int_{R_{max}-a}^{R_{max}} dr\, P(r,H) \sim \left(\frac{a}{R_{max}}\right)^{1-\nu}$$

$$\Rightarrow \quad \frac{dR_{max}}{dH} \sim \frac{dR}{dH} \sim \left(\frac{a}{R}\right)^{1-\nu} \qquad (R_{max} \sim R)$$

$$\Rightarrow \quad H \sim R^{2-\nu} \quad \Rightarrow \quad D = 2-\nu = \frac{3\pi - \phi}{2\pi - \phi}$$

In particular $\phi \geq 0 \Rightarrow D \geq 3/2$

rigorous proof by H. Kesten (1987)

Summary: $d = 2$: $\frac{3}{2} \leq D \leq 2$

$d > 2$: $d-1 \leq D \leq d$