3.3 Laplacian fractals & Laplacian instabilities

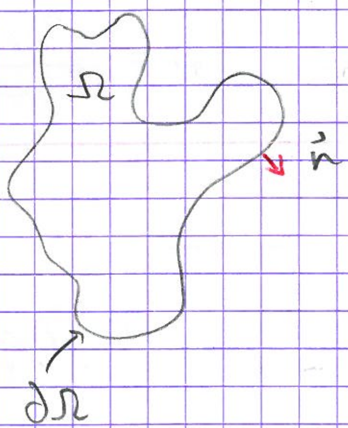
a) Continuum limit of DLA

DLA is a discrete, stochastic growth process.

- **Deterministic limit:** Noise reduction (on-lattice)

  $\hat{=}$ growth occurs only when a site has been hit by m walkers, $m \to \infty$ Bild

- **Continuum limit:** Represent cluster by a domain $\Omega \in \mathbb{R}^d$ with boundary $\partial\Omega$



$u(\vec{r})$: density / occupation probability of RW's

$\vec{r} \in \partial\Omega$: $u = 0$ (i)

$r \in \vec{\mathbb{R}}^d / \Omega$: $\nabla^2 u = 0$ (ii)

u is a Laplacian field

The growth probability $P(\vec{r})$ at a point $\vec{r} \in \partial\Omega$ is

$$P(\vec{r}) \sim \vec{n} \cdot \nabla u \qquad \vec{n}: \text{outward normal}$$

$$\int_{\partial\Omega} P(\vec{r})\, ds = 1$$

$P(\vec{r})$ is the harmonic measure on $\partial\Omega$.

In the deterministic limit $P \to v_n$

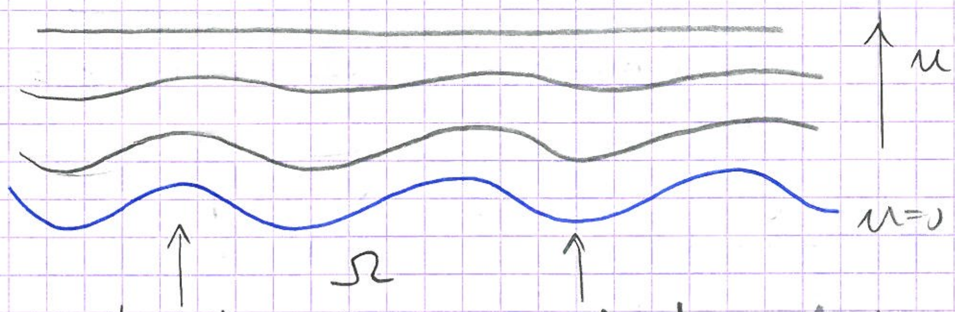
(iii) $v_n = \vec{n} \cdot \nabla u$ normal velocity of the boundary

Together the eq. (i) – (iii) define the basic Laplacian moving boundary value problem. 18.1.07

Planar fronts evolving under (i) – (iii) are unstable.

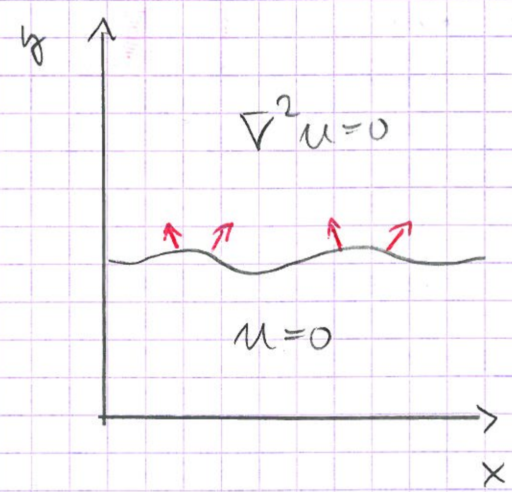
24.1.08

Qualitative:



$\Rightarrow$ crowding of lines of constant u at protrusions of $\partial\Omega$ implies preferential growth of protrusion $\Rightarrow$ instability

Quantitative: Linear stability analysis (d=2)



Front position $y = h(x,t)$

Planar front: $y = vt$

Laplacian field:

$$u_0(x,y,t) = \begin{cases} v(y-vt) & y > vt \\ 0 & y < vt \end{cases}$$

Consider a small perturbation of wave number q:

$$\left.\begin{aligned} h(x,t) &= vt + \hat{\varepsilon}_h(q,t)\, e^{iqx} \\ u(x,y,t) &= u_0(x,y,t) + \hat{\varepsilon}_u(q,t)\, e^{-(y-vt)/\lambda}\, e^{iqx} \end{aligned}\right\}$$

(ii) $\nabla^2 u = 0$ $\Rightarrow$ $q^2 - \frac{1}{\lambda^2} = 0$

$\Rightarrow$ $\lambda = 1/|q|$ penetration depth of the perturbation

(i), (iii) $\Rightarrow$ $$\frac{d\hat{\varepsilon}_u}{dt} = -\frac{1}{\lambda}\,\hat{\varepsilon}_u = v\,|q|\,\hat{\varepsilon}_u$$ Problem

$\Rightarrow$ perturbation grows at rate $\omega(q) = v\,|q|$

The growth diverges for $|q| \to \infty$
$\Rightarrow$ perturbations of arbitrarily short wavelength grow arbitrarily fast

$\Rightarrow$ problem is mathematically ill-posed

Consequences:

(i) Dynamics generates singularities from smooth initial conditions in finite time (Shraiman, Bensimon, 1984) Bild

$\Rightarrow$ cf. shock formation, Sect. I.2.4

(ii) Any physical realization must include a regularization at short length scales.

(iii) In DLA the regularization is due to the finite size of particles
$\Rightarrow$ model generates structure down to the particle (lattice) scale

Laplacian boundary value problems with regularization:

- solidification
- fluid flow in porous media
- dielectric breakdown

} to be considered in the following

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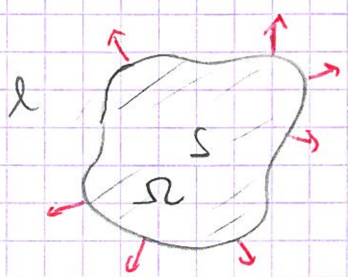
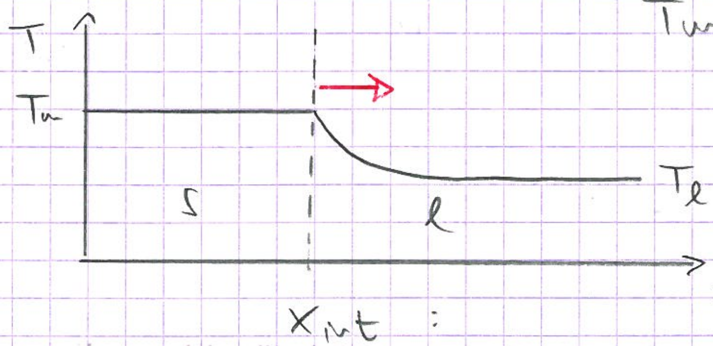
b) The Mullins-Sekerka instability

Solidification: Growth of a solid into an undercooled melt





T_m: melting point

x_{int}: solid-liquid interface

Rate limiting process: Diffusion of latent heat away from the interface

$$\Rightarrow \quad \frac{\partial T(\vec{r},t)}{\partial t} = D_T \nabla^2 T \simeq 0 \quad \text{outside of } \Omega \qquad \text{(ii)}$$

$\Rightarrow$ $T(\vec{r},t)$ is the Laplacian field.

Boundary condition at $\partial\Omega$:

Energy conservation $\Longleftrightarrow$ production of latent heat = heat current away from $\partial\Omega$

$$\Rightarrow \quad \underline{Q_L v_n = - D_T c (\nabla T \cdot \vec{n})} \qquad \text{(iii)}$$

c : specific heat , Q_L : latent heat

The boundary condition (i) for T is modified by the Gibbs-Thomson effect:

At a curved interface (see II.5.4)

$$\mu = \mu_{se} \rightarrow \mu_{se} + \Delta\mu, \quad \Delta\mu = \Omega \frac{2\gamma}{R} = \Omega \gamma \varkappa$$

Ω : atomic volume, $\varkappa$: curvature

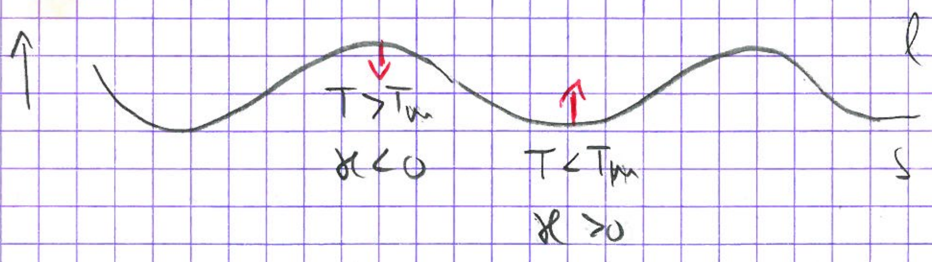
$$\Rightarrow \quad T = T_m \rightarrow T_m + \Delta T,$$

$$\Delta T = \frac{\partial T}{\partial \mu} \Delta\mu = - \frac{T_m}{Q_L} \gamma \varkappa = - T_m \cdot d_0 \varkappa$$

$$\Rightarrow \quad T\Big|_{\partial\Omega} = T_m (1 - d_0 \varkappa) \qquad \text{(i)}$$

$d_0 = \gamma / Q_L$: capillary length

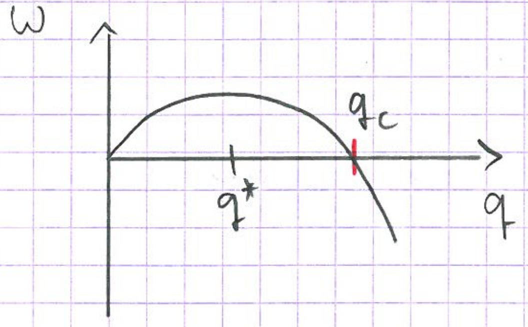
Surface tension smoothens the interface:



Linear stability analysis with (i):

$$\underline{\omega(q) = v|q|(1 - d_0 l_D q^2)} \qquad \text{(Mullins, Sekerka, 1964)}$$

$l_D = 2D_T/v$ diffusion length



$\Rightarrow$ stability ($\omega < 0$) for large q

$\Rightarrow$ surface tension regularizes the problem and makes it well-posed.

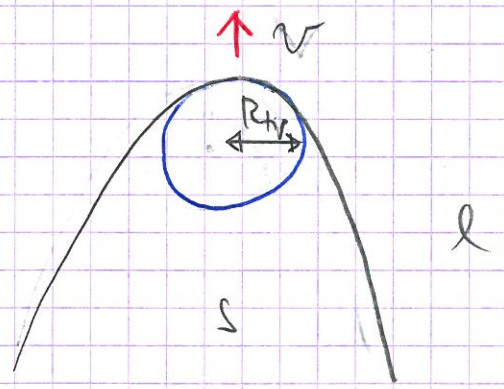
Characteristic length scale: $q_c = \sqrt{d_0 l_D}^{\,-1}$

(cf. II.5.4) $\lambda^* = \frac{2\pi}{q^*} = 2\pi\sqrt{3}\sqrt{d_0 l_0}$

d_0 : microscopic ($\sim$ Å)
l_D : macroscopic ($\sim$ mm)
$\Rightarrow \lambda^*$: mesoscopic (μm)

Nonlinear evolution: Dendrites Bild

$\gamma = 0$: Ivantsov parabola (1947)



$\Rightarrow \frac{R_{tip}}{l_D} = \frac{R_{tip}\, v}{2D} = \text{const.}$

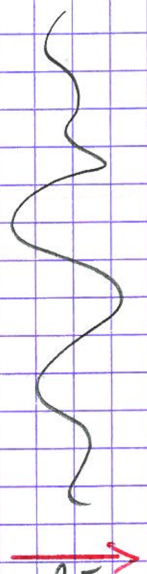
$\gamma > 0$: Unique selection of R_{tip} & v requires crystal anisotropy of γ

$\Rightarrow$ microscopic solvability theory (~1986 – 1990)

c) The Saffman-Taylor instability

Two-fluid flow in a porous medium (rock, sand):

1   2



P_1 P_2 : pressure

$\vec{v}_1$ $\vec{v}_2$: flow velocity

η_1 η_2 : dynamic viscosity

v

e.g.: H_2O oil

Darcy's law: (1856)

Bild

$$\vec{v}_\alpha = -\frac{k}{\eta_\alpha}\nabla P_\alpha \qquad \alpha = 1,2$$

k: permeability

Incompressibility:

$$\frac{\partial \rho_\alpha}{\partial t} = -\nabla\cdot(\rho_\alpha \vec{v}_\alpha) = 0$$

$$\Rightarrow \nabla\cdot\vec{v}_\alpha = 0$$

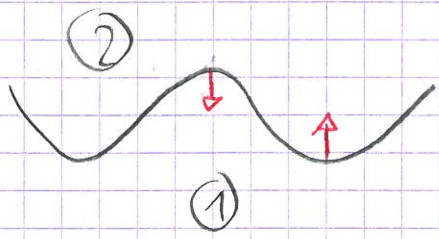
$$\Rightarrow \nabla^2 P_\alpha = 0 \qquad \text{(ii)}$$

$\Rightarrow$ fluid pressure is a "Laplacian field".

Interface motion: $\vec{v}_1 = \vec{v}_2$ at the interface

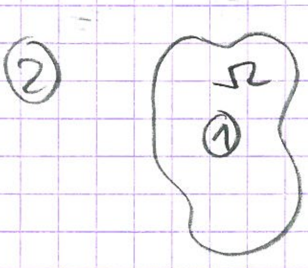
$$\Rightarrow v_n = \vec{v}_1\cdot\vec{n} = \vec{v}_2\cdot\vec{n} = -\frac{k}{\eta_{1,2}}(\vec{n}\cdot\nabla P_{1,2}) \qquad \text{(iii)}$$

Boundary condition for the pressure:



$$(P_1 - P_2)\Big|_{\partial\Omega} = \frac{2\gamma}{R} = \gamma\varkappa \quad (\text{I})$$

Special case: $\eta_1 \ll \eta_2 \equiv \eta$ (e.g. air into H_2O)



$$\Rightarrow \left.\begin{aligned} &P = P_1 = \text{const.} : \Omega \\ &\nabla^2 P = 0 \quad \text{in } \mathbb{R}^d / \Omega \\ &P\big|_{\partial\Omega} = P_1 - \gamma\varkappa \\ &v_n = -\frac{k}{\eta_2} \nabla P \cdot \vec{n} \end{aligned}\right\}$$

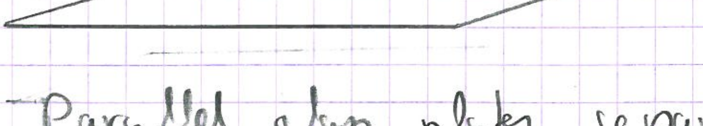
$$\left.\begin{aligned} \eta_1 &\to 0 \\ \nabla P_1 &\to 0 \end{aligned}\right\} \; \vec{v}_1\Big|_{\partial\Omega} = \vec{v}_2\Big|_{\partial\Omega}$$

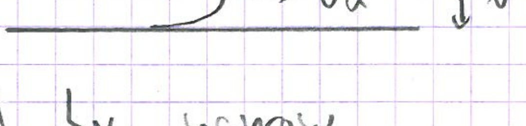
$\Rightarrow$ regularized Laplacian boundary value problem, equivalent to solidification.

Additional complication:

Spatial randomness of the porous medium

$\Rightarrow$ **Model problem: Hele-Shaw cell** (1898)



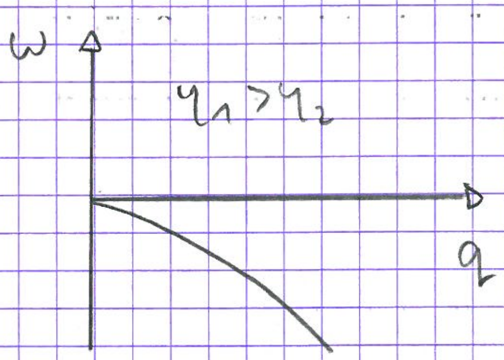


Parallel glass plates separated by narrow gap $\Rightarrow$ $k = b^2/12$, two-dimensional flow

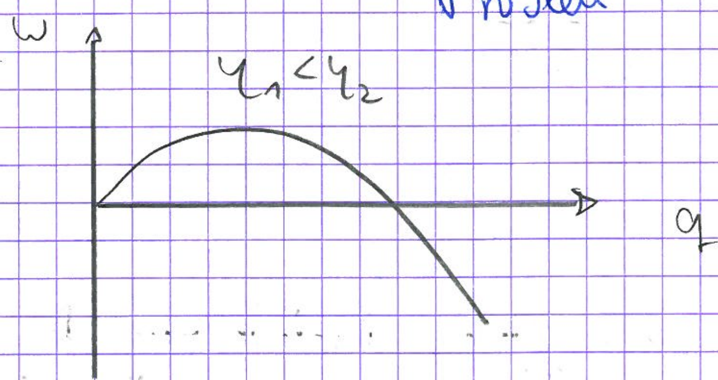
Linear stability analysis: (Saffman & Taylor, 1958)

$$\omega(q) = -|q|\left(v\cdot\frac{\eta_1-\eta_2}{\eta_1+\eta_2} + \frac{k\gamma q^2}{\eta_1+\eta_2}\right)$$

Problem

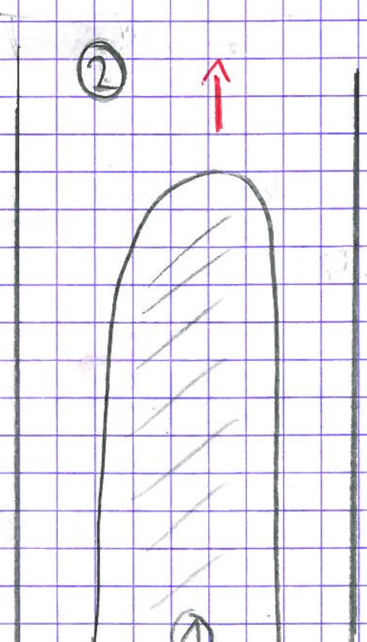


stable



unstable, well-posed

Nonlinear evolution: Fingers in a channel



$\gamma = 0$: Fingers of any width W exist
(Saffman & Taylor, 1958)

$\gamma > 0$: Selection of unique value of W/L

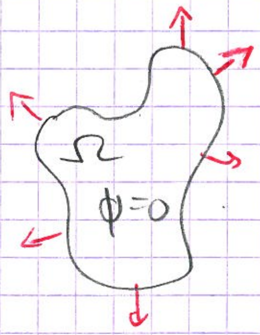
With increasing $Ca = \frac{\eta v}{\gamma}$

$\Rightarrow W/L \to \frac{1}{2}$

Bilder

d) Dielectric breakdown

Propagation of a conducting, ionized region in an insulating medium:



$\nabla^2 \phi \Rightarrow$ electrostatic potential ϕ is the Laplacian field

Examples: Lichtenberg figures, lightnings,... Bilder

Dynamics?: $v_n \sim |\nabla \phi \cdot \vec{n}|^{\eta}$, $\eta > 0$

(Niemeyer et al, 1984)

Implementation: (i) Solve $\nabla^2 \phi = 0$ on the lattice

(ii) Add boundary sites to Ω with probability $\sim (\nabla \phi)^{\eta}$

Alternative: RW-simulations to solve $\nabla^2 \phi = 0$

$$\left.\begin{array}{ll} \eta = 1: & \text{DLA} \\ \eta \to \infty: & D \to 1 \\ \eta \to 0: & D \to d \end{array}\right\} \quad \text{Bild}$$

Regularization?