4° Turbulence

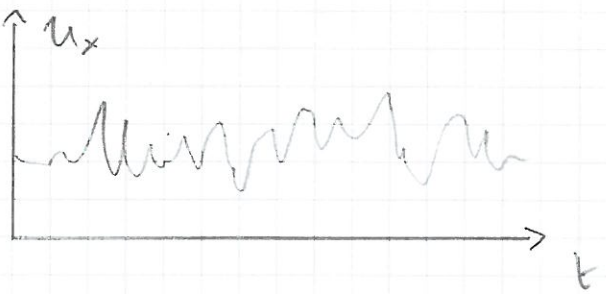
4.1. Phenomenology

- Incompressible fluids that are strongly driven develop a random, scale-invariant structure ("whorls within whorls") on scales smaller than the scale of driving (integral scale L)

Example: Grid turbulence

⊗ probe recording velocity component $u_x(t)$, $u_y(t)$

$\Rightarrow$ self-affine random signal



$$\langle (u_x(t+\tau) - u_x(t))^2 \rangle \sim |\tau|^{\zeta_2}$$

- Taylor hypothesis: Temporal signal $u_{x,y}(t)$ reflects the spatial variation of the velocity field which is advected by the mean flow:

$$\langle (u_x(x+r) - u_x(x))^2 \rangle \sim |r|^{\zeta_2}$$

structure function

- Energy cascade: Energy is injected into the fluid at the integral scale and dissipated at the dissipation scale $\ell \ll L$. In the inertial range $\ell \ll |\vec{r}| \ll L$ energy is transferred from large to small scales but not dissipated. This results in a power law energy spectrum

$$\mathcal{E}(k) \sim k^{-(1+2\zeta_2)}$$

$\mathcal{E}(k)$: energy content of velocity fluctuations of wave vector k

Total energy is $\int_0^\infty dk\, \mathcal{E}(k)$

4.2. Navier-Stokes equation

$\vec{u}(\vec{r},t)$: Fluid velocity

Total time derivative:

$$\frac{d}{dt}\vec{u}(\vec{r}(t),t) = \frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}$$

advection term, non linear

Newton's law of motion:

$$\rho_0 \frac{d\vec{u}}{dt} = \underbrace{\rho_0}_{\text{constant density (incompressible)}} \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} \right) = - \underbrace{\nabla P}_{\text{pressure}} + \underbrace{\eta \nabla^2 \vec{u}}_{\text{internal friction}}$$

$$\Rightarrow \quad \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho_0}\nabla P + \nu \nabla^2\vec{u} + \frac{1}{\rho_0}\vec{F}_{ext}$$

Navier-Stokes eq. $\qquad$ kinematic viscosity $\nu = \eta/\rho_0$

- Incompressibility: $\dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$

  $\rho \equiv \rho_0 \quad \Rightarrow \quad \nabla\cdot\vec{u} = 0$

This constraint can be used to eliminate the pressure:

$$0 = \frac{\partial}{\partial t}(\nabla\cdot\vec{u}) = -\nabla\cdot(\vec{u}\cdot\nabla)\vec{u} - \frac{1}{\rho_0}\nabla^2 P$$

$$\Rightarrow \quad \nabla^2 P = -\rho_0 \nabla\cdot(\vec{u}\cdot\nabla)\vec{u} \qquad \text{Poisson eq.}$$

$$\Rightarrow \quad P = -\rho_0 \nabla^{-2}\left(\nabla\cdot(\vec{u}\cdot\nabla)\vec{u}\right)$$

$$\Rightarrow \quad \frac{\partial \vec{u}}{\partial t} + (\mathbb{1} - \underbrace{\nabla\nabla^{-2}\nabla\cdot}_{\text{non-local}})(\vec{u}\cdot\nabla)\vec{u} = \nu\nabla^2\vec{u}$$

$\Rightarrow$ P is not an independent dynamic variable.

- **Reynolds number:**

  - driving scale L
  - typical velocity scale u_0
  - viscosity ν with $[\nu] = [x]^2/[t]$

$\Rightarrow$ dimensionless combination $Re = \dfrac{L u_0}{\nu}$

  - Similarity: Fluids with the same Re are equivalent
  - Strong driving implies $Re \gg 1$; fully developed turbulence is the limit $Re \to \infty$.

- **Energy dissipation:** Energy density $\sim \frac{1}{2}|\vec{u}|^2$

$$\Rightarrow \frac{d}{dt}\left\langle \frac{1}{2}|\vec{u}|^2 \right\rangle = -\underbrace{\frac{1}{2}\nu\left\langle \sum_{ij}\left|\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right|^2\right\rangle}_{=:\langle \varepsilon(\vec{r},t)\rangle}$$

Key idea: Turbulent fluid maintains finite $\langle\varepsilon\rangle$ at $\nu \to 0$

$\Rightarrow$ $\vec{u}(\vec{r},t)$ is non-differentiable

4.3. Kolmogorov's 1941 theory (K41)

<u>Goal:</u> Quantify scenario of a scale-invariant energy cascade

a) Scale-invariance of the Euler equation

Since dissipation is irrelevant in the inertial range, consider the Euler eq. with $\nu = 0$:

$$\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho_0}\nabla P$$

<u>Rescaling of $\vec{u}, \vec{r}, t$</u>: (cf. KPZ-eq.)

$$\vec{u}'(\vec{r},t) = b^{-\mathcal{S}}\,\vec{u}(b\vec{r}, b^z t)$$

$$\Rightarrow \frac{\partial\vec{u}'}{\partial t} = b^{z-\mathcal{S}}\frac{\partial\vec{u}}{\partial t}$$

$$\vec{u}'\cdot\nabla\vec{u}' = b^{-2\mathcal{S}+1}\,\vec{u}\cdot\nabla\vec{u}$$

$\Rightarrow$ scale invariance requires $z + \mathcal{S} = 1$

(KPZ: $z + \alpha = 2$)

$\Rightarrow$ with $z = 1 - \mathcal{S}$, the Euler eq. is scale invariant for <u>any</u> choice of $\mathcal{S}$.

To fix $\mathcal{S}$, Kolmogorov assumes that the energy dissipation ε is independent of scale.

b) Constant energy dissipation

We first examine how ν changes under rescaling:

$$\left.\frac{\partial \vec{u}'}{\partial t}\right|_{diss} = b^{\overbrace{2-\zeta}^{1-2\zeta}} \left.\frac{\partial \vec{u}}{\partial t}\right|_{diss} = b^{1-2\zeta}\, \nu\, \nabla^2 \vec{u} =$$

$$= \underbrace{b^{1-2\zeta}\, \nu\, b^{\zeta-2}}_{= \nu' = b^{-(1+\zeta)}\nu} \nabla'^2 \vec{u}'$$

$\Rightarrow$ ν decreases on larger scales provided $\zeta > -1$.

The rescaled energy dissipation is

$$\varepsilon' = \frac{1}{2}\nu' \underbrace{\sum_{ij} \left| \frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'} \right|^2}_{= b^{2-2\zeta} \sum_{ij} \left| \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right|^2} = b^{-(1+\zeta)}\, b^{2-2\zeta}\, \varepsilon$$

$$= b^{1-3\zeta}\, \varepsilon$$

$\Rightarrow$ $\varepsilon' = \varepsilon$ requires $\boxed{\zeta = 1/3}$

c) Structure functions

The scaling exponent ζ is analogous to the roughness exponent α of an interface; in particular

$$S_2(\vec{r}) := \langle |\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})|^2 \rangle \sim |\vec{r}|^{2\zeta} = |\vec{r}|^{2/3}$$

In turbulence one considers also higher order structure functions

$$S_q(\vec{r}) = \langle |\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})|^q \rangle$$

Scale invariance implies

$$S_q(b\vec{r}) = \langle |\vec{u}(b(\vec{x}+\vec{r})) - \vec{u}(\vec{x})|^q \rangle$$

$$= b^{q\zeta} \langle |\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})|^q \rangle = b^{q\zeta} S_q(\vec{r})$$

and setting $b|\vec{r}| = 1 \Rightarrow S_q(\vec{r}) \sim |\vec{r}|^{q\zeta} = |\vec{r}|^{q/3}$

Moreover, since the dimension of ε is $\frac{[l^2]}{[t]^3}$, $(\varepsilon|\vec{r}|)^{1/3}$ has the dimension of a velocity

$$\Rightarrow \quad S_q(\vec{r}) \sim |\varepsilon \vec{r}|^{q/3}$$

S

Exact relation for $q = 3$: $$S_3(\vec{r}) = \frac{4}{5}\,\varepsilon|\vec{r}|$$

(4/5 - law)

d) Energy spectrum

Self-affinity of $\vec{u}(\vec{r})$ implies that the spatial Fourier spectrum behaves as

$$\langle|\hat{\vec{u}}(\vec{k})|^2\rangle \sim |\vec{k}|^{-(d+2\zeta_2)} = |\vec{k}|^{-\underbrace{(3+2\zeta_2)}_{=3+2/3}}$$

By isotropy, the total energy is

$$E_{tot} = \frac{1}{2}\int d\vec{k}\,|\hat{\vec{u}}(\vec{k})|^2 =$$

$$= \frac{1}{2}\cdot 4\pi\int_0^\infty dk\,k^2|\hat{\vec{u}}(\vec{k})|^2 = \int_0^\infty dk\,\mathcal{E}(k)$$

$$\Rightarrow \quad \mathcal{E}(k) \sim \varepsilon^{2/3}\,k^{-5/3}$$ 5/3- law

e) The dissipation scale

- The fact that $\zeta < 1$ implies that the velocity field is not differentiable in the inertial range; rather, it is Hölder continuous with Hölder-exponent $\zeta = 1/3$.

- At very short scales ($|\vec{r}| \to 0$) one expects that $\vec{u}$ becomes differentiable due to dissipation. This implies that

$$S_q^{diss}(\vec{r}) = \langle |\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})|^q \rangle \simeq \langle |\nabla \vec{u}|^q \rangle |\vec{r}|^q$$

Using $\varepsilon = \frac{1}{2} \nu \sum_{ij} \left| \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right|^2 \sim \nu |\nabla u|^2$

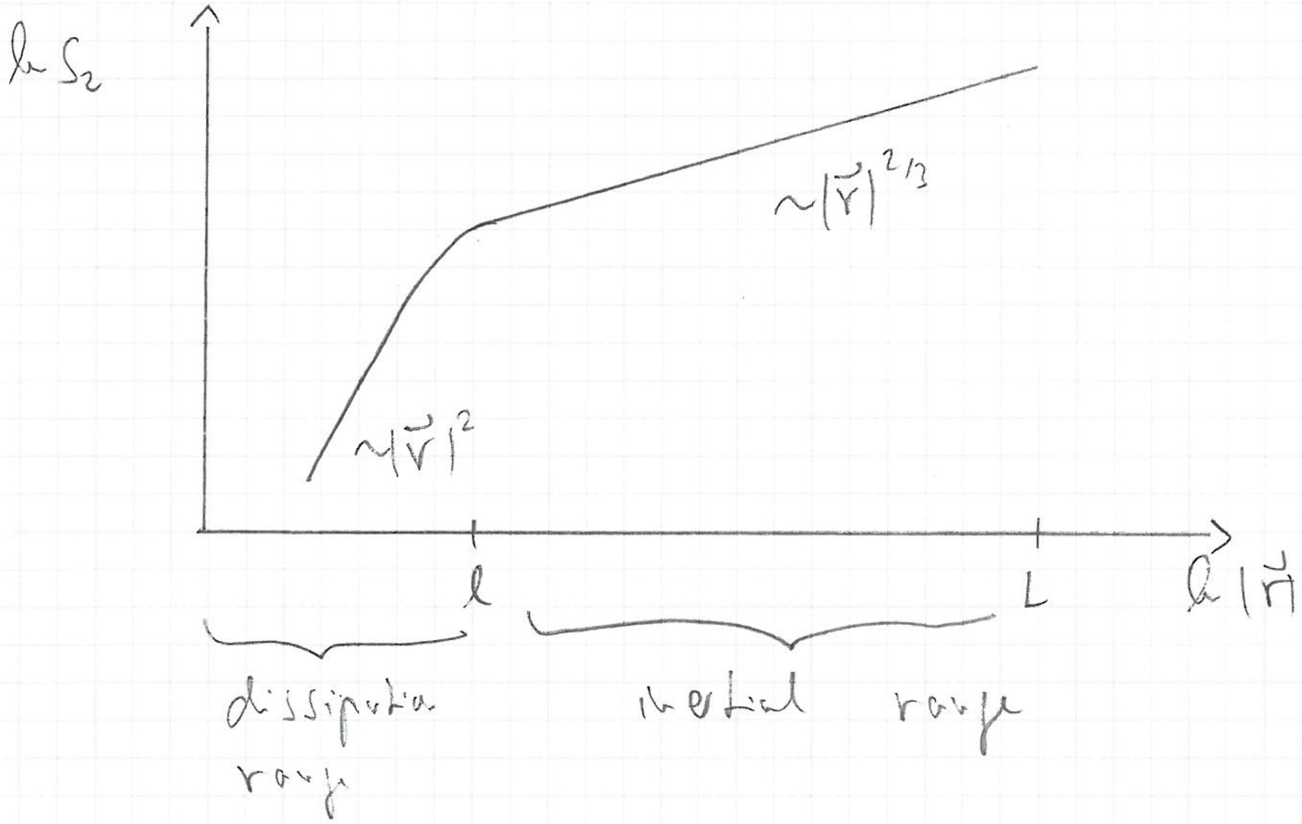
we see that

$$|\nabla u| \sim \sqrt{\varepsilon/\nu}, \quad |\nabla u|^q \sim (\varepsilon/\nu)^{q/2}$$

$$\Rightarrow \quad S_q^{diss}(\vec{r}) \sim \left(\frac{\varepsilon}{\nu}\right)^{q/2} |\vec{r}|^q$$

Matching this to the inertial range result (S) at the dissipation scale ℓ yields

$$\left(\frac{\varepsilon}{\nu}\right)^{q/2} \ell^q \sim (\varepsilon \ell)^{q/3} \quad \Rightarrow \quad \ell \sim \left(\nu^3/\varepsilon\right)^{1/4}$$



The ratio L/l increases with Re:

- Velocity scale $u_0 \sim \sqrt{S_2(L)} \sim (\varepsilon L)^{1/3}$
- Length scale L

$$\Rightarrow \quad Re = \frac{u_0 L}{\nu} = \frac{\varepsilon^{1/3} L^{4/3}}{\nu} = \left(\frac{L}{l}\right)^{4/3}$$

$$\Rightarrow \quad \underline{L/l \sim Re^{3/4}}$$

4.4. Intermittency

- Experimentally one observes deviations from K41 in that the exponents ζ_q of the structure functions vary non-linearly with q:

$$S_q(\vec{r}) \sim |\vec{r}|^{\zeta_q} \quad \text{with} \quad \zeta_q \begin{cases} > q/3 & q < 3 \\ < q/3 & q > 3 \end{cases}$$

- These deviations reflect the strong (intermittent) fluctuations of the energy dissipation ε, which is not well represented by its average:

$$\langle \varepsilon^n \rangle \gg \langle \varepsilon \rangle^n$$

for large Re; more precisely, one expects that

$$\frac{\langle \varepsilon^n \rangle}{\langle \varepsilon \rangle^n} \sim \left(\frac{L}{\ell} \right)^{\mu_n} \quad \text{with} \quad \begin{aligned} &\mu_1 = 0 \\ &\mu_n > 0 \text{ for } n > 1 \end{aligned} \tag{ε}$$

- To see how this affects the structure functions, we sketch the refined similarity hypothesis due to Obukhov and Kolmogorov (1962).

- Introduce the energy dissipation on scale r

$$\varepsilon_r := r^{-3} \int_{V_r} d^3\vec{x}\; \varepsilon(\vec{x})$$

where $l < r < L$ and V_r is a volume of radius r.

- Then postulate that ε generalizes to

$$\frac{\langle \varepsilon_r^n \rangle}{\langle \varepsilon_r \rangle^n} \sim \left(\frac{L}{r}\right)^{\mu_n}$$

and apply Kolmogorov's hypothesis in the generalized form

$$S_q(\vec{r}) \sim \langle (\varepsilon_r r)^{q/3} \rangle \sim \langle \varepsilon_r^{q/3} \rangle r^{q/3}$$

$$\sim \langle \varepsilon \rangle^{q/3} \left(\frac{L}{r}\right)^{\mu_{q/3}} r^{q/3}$$

$$\sim \langle \varepsilon \rangle^{q/3} L^{\mu_{q/3}} r^{\zeta_q}$$

with $\zeta_q = \frac{q}{3} - \mu_{q/3} < \frac{q}{3}$ for $q > 3$

Remarks:

- In contrast to K41, the structure functions depend on the integral scale L through the factor $L^{-\mu_{q/3}}$. This is unavoidable on dimensional grounds.
- The problem has been shifted from the prediction of the ζ_q to the prediction of the μ_q. Several (more or less phenomenological) models have been proposed:

β-model: Total volume of daughter eddies reduces by factor β in each step of the cascade

$$\Rightarrow \quad \zeta_q = \frac{q}{3} + (3-D)\left(1-\frac{q}{3}\right), \quad D = 3 - \frac{\ln\beta}{\ln r}$$

r: cascade scale factor

log-normal: Cascade with random multiplicative factors w/ log-normal distribution

$$\Rightarrow \quad \zeta_q = \frac{q}{3} + \frac{\mu}{18}\left(3q - q^2\right)$$

μ: free parameter

She-Leveque: Coherent structure

$$\Rightarrow \quad \zeta_q = \frac{q}{9} + 2 - 2\left(\frac{2}{3}\right)^{q/3}$$