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Problems in Advanced Statistical Physics

Problem 27: The Glauber-Ising chain

The transition rate for a spin flip in the one-dimensional Glauber-Ising model at zero external field is defined by[1]

$$\Gamma(\sigma_i \to -\sigma_i) = \frac{1}{2}[1 - \frac{\gamma}{2}\sigma_i(\sigma_{i+1} + \sigma_{i-1}]. \tag{1}$$

a.) Determine the coefficient γ such that the rates (1) satisfy detailed balance with respect to the Ising energy.

b.) Consider a chain with periodic boundary conditions. Use the master equation to derive the closed set of equations

$$\frac{dG_{i,j}}{dt} = -2G_{i,j} + \frac{\gamma}{2}[G_{i+1,j} + G_{i-1,j} + G_{i,j+1} + G_{i,j-1}] \tag{2}$$

for the equal time spin-spin correlation function $G_{i,j}(t) = \langle \sigma_i(t)\sigma_j(t) \rangle$, where $i \neq j$. For $i = j$ we obviously have $G_{i,i}(t) = 1$ independent of t.

c.) If the ensemble of initial conditions is translationally invariant, so that $G_{i,j}$ only depends on $r = |i - j|$, then this property is preserved under the dynamics, and (2) reduces to

$$\frac{dG_r}{dt} = -2G_r + \gamma[G_{r+1} + G_{r-1}]. \tag{3}$$

Show that the stationary solution of (3) agrees with the equilibrium pair correlation function for the Ising chain.

d.) Now consider (3) at zero temperature[2]. To extract the behavior of the correlations at long times and large distances, approximate the difference equation (3) by a partial differential equation. Show that this equation has a solution of the scaling form

$$G(r,t) = \mathcal{G}(r/t^n),$$

and determine the domain growth exponent n as well as the scaling function $\mathcal{G}$. Show that the behavior of the scaling function near the origin is consistent with *Porod's law*, which states that the scaling function $\mathcal{G}$ has a cusp singularity at zero, $1 - \mathcal{G}(x) \sim |x|$ for $x \to 0$.

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[1] R.J. Glauber, J. Math. Phys. **4**, 294 (1963).
[2] A.J. Bray, J. Phys. A **22**, L67 (1989).

e.) The two-time correlation function $G_r^{(2)}(t,t') = \langle \sigma_i(t)\sigma_{i+r}(t')\rangle$ satisfies an equation similar to (3), which reads at $T = 0$

$$\frac{dG_r^{(2)}}{dt} = -G_r^{(2)} + \frac{1}{2}(G_{r+1}^{(2)} + G_{r-1}^{(2)}), \tag{4}$$

and it is assumed that $t > t'$. As in part d.), solve (3) in the continuum approximation using the initial condition $G^{(2)}(r,t',t') = G(r,t')$. Evaluate the resulting expression for $r = 0$, and thus obtain the autorcorrelation function

$$A(t,t') = G^{(2)}(0,t,t') = \frac{2}{\pi}\sin^{-1}\left(\sqrt{\frac{2t'}{t+t'}}\right). \tag{5}$$

Show that $A(t,t') \sim (t'/t)^{\lambda n}$ for $t \gg t'$, and find the autocorrelation exponent λ.

Problem 28: Droplet dynamics in the Allen-Cahn equation

Here we want to derive the dynamics of a shrinking droplet directly from the Allen-Cahn equation for the order parameter. The droplet is described by a radially symmetric solution $\phi(r,t)$ of the form

$$\phi(r,t) = \Phi(r - R(t)), \tag{6}$$

where $\Phi(x \to \pm\infty) = \pm\phi_0$, and the transition from $-\phi_0$ to ϕ_0 occurs on the scale ξ around $x = 0$. Insert (6) this into the d-dimensional Allen-Cahn equation expressed in polar coordinates, and show that in the limit $R \gg \xi$, where $\Phi'(x)$ reduces to a δ-function at $x = 0$, the droplet radius satisfies the equation

$$\frac{dR}{dt} = -\frac{d-1}{R}$$

as announced in the lectures.

Problem 29: Zipf's law for random texts

Zipf's law states that the number $N(x)$ of distinct words that occur with frequency x in a text is proportional to $x^{-\alpha}$, where $\alpha \approx 2$. Here we consider random texts, which are random uncorrelated sequences consisting of m different letters and one space sign which separates different words. All letters occur with equal probability q, and the space sign with probability $q_s = 1 - mq$. Clearly in this model all words of the same length l occur with the same probability. Show that both the probability of occurrence of a given word, and the number of distinct words of length l depend exponentially on l, and deduce from this the power law $N(x) \sim x^{-\alpha}$. Investigate the behavior of α for large m and small q_s.