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Problems in Advanced Statistical Physics

Problem 5: Generating functions, cumulants, and the central limit theorem

Let $X_1, X_2, \ldots$ be independent and identically distributed (iid) random variables with common probability density $p(x)$. The *generating function* $\varphi(k)$ with real k is defined as

$$\varphi(k) = \langle e^{ikX_i} \rangle = \int_{-\infty}^{\infty} dx\ e^{ikx} p(x). \tag{1}$$

The integral is absolutely convergent (because $|e^{ikx}| = 1$ and $\int p(x)dx = 1$), $\varphi(k)$ is continuous, and $|\varphi(k)| \le 1$ for all real k. Frequently, $\varphi(k)$ is called *characteristic function.*

a.) Calculate $\varphi(k)$'s for the following distributions:

$$\text{Gaussian: } p(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \qquad \text{Cauchy: } p(x) = \pi^{-1}(1+x^2)^{-1}$$

$$\text{Binomial: } p(x) = \sum_{n=0}^{N} \delta(x-n) \binom{N}{n} \rho^n (1-\rho)^{N-n}$$

$$\text{Poisson: } p(x) = \sum_{n=0}^{\infty} \delta(x-n) \frac{\lambda^n}{n!} e^{-\lambda}.$$

Show using the generating functions that the binomial distribution approaches the Poisson distribution as $\rho \to 0$ and $N \to \infty$ with $\lambda = \rho N$ fixed.

b.) The nth moment is defined as $m_n \equiv \int_{-\infty}^{\infty} x^m p(x) dx$, provided the integral exists. Prove the relation

$$m_n = (-i)^m \frac{d^m \varphi(k)}{dk^m}\bigg|_{k=0}. \tag{2}$$

This is the reason why we call $\varphi(k)$ the (moment) *generating* function. Does the Cauchy distribution satisfy Eq. (2)?

c.) $\chi(k) \equiv \ln(\varphi(k))$ is called the *cumulant generating function* and has an expansion

$$\chi(k) = \sum_{n=1}^{\infty} \kappa_n \frac{(ik)^n}{n!}, \tag{3}$$

where κ_n is called the n-th cumulant. Prove that

$$\kappa_n = m_n - \sum_{p=1}^{n-1} \binom{n-1}{p} \kappa_p m_{n-p}.$$

Write down m_n as a function of κ_m and vice versa up to $n = m = 4$. What is κ_n for the Gaussian?

d.) Let

$$S_N = \frac{(X_1 - \mu) + \cdots + (X_N - \mu)}{\sqrt{N}\sigma},$$

where μ and σ^2 are the mean and the variance of $p(x)$ ($\mu < \infty$, $\sigma < \infty$). What is the generating function for S_N in terms of $\varphi(k)$ of Eq. (1)? Show that as $N \to \infty$, this generating function approaches the generating function of the Gaussian distribution in a.). This statement is called the *central limit theorem*. Show that the binomial and Poisson distributions become Gaussian but the Cauchy distribution does not.

Hint: Finite variance implies the existence of the second derivative of $\varphi(k)$ for all k. Perform a Taylor expansion up to k^2 and then take the $N \to \infty$ limit.

Problem 6: A Lévy flight on living polymers

Living polymers are linear chains of macromolecular aggregates in dilute solution, which break up and recombine on a time scale τ^*. From these processes a stationary distribution of polymer length l results, which is of the form

$$P(l) = l_0^{-2}\, l\, e^{-l/l_0}, \tag{4}$$

with l_0 denoting the mean length.

The diffusion of a fluorescent tracer particle in the polymer solution can be described as follows[1]: The particle spends a time τ^* on a polymer of length l, and diffuses with the corresponding diffusion constant $D(l)$, which depends on l according to

$$D(l) = D_0(l_D/l)^\alpha \tag{5}$$

with $\alpha \approx 2.15$. When the polymer breaks up or recombines after a time τ^*, the particle continues to diffuse on a polymer with a new length, which is chosen randomly from the distribution (4). The particle thus performs a random flight with variable jump length r. Compute the probability distribution of r using (4) and (5). How large must α be for the resulting motion of the particle to be *superdiffusive*, in the sense that the mean square displacement grows faster than linearly with the number of steps?

[1] A. Ott, J.P. Bouchaud, D. Langevin, W. Urbach: Phys. Rev. Lett. **65**, 2201 (1990).