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Problems in Advanced Statistical Physics

Problem 7: Equal probability of microstates and Bernoulli measure

For the one-dimensional asymmetric exclusion process with N particles on a ring of L sites it has been shown that all $\binom{L}{N}$ microstates are equally likely in the nonequlibrium stationary state. Prove that this implies *Bernoulli measure* in the limit $L, N \to \infty$ at fixed density $\rho = N/L$, which means that in the infinite system each site is independently occupied or vacant with probability ρ and $1-\rho$, respectively. To this end, compute correlation functions $\langle \eta_i \eta_j \rangle$, $\langle \eta_i \eta_j \eta_k \rangle$ etc. for the finite system, and take the limit $L, N \to \infty$. Also compute the leading finite size correction to the stationary particle current.

Problem 8: Single file diffusion

Here we want to establish the anomalous diffusion law[1] $\langle X_0(t)^2 \rangle \sim t^{1/2}$ for a tagged particle in the one-dimensional Langmuir lattice gas, where $X_0(t)$ denotes the position of a particle that resides at the origin at time $t = 0$. We work in the continuum setting. The key relation is

$$X_0(t) \approx \frac{1}{\bar{\rho}} \int_0^t ds\, j(0, s) \tag{1}$$

for long times, where $\bar{\rho}$ is the mean density in the system and $j(0, t)$ is the particle current through the origin at time t.

We decompose the density $\rho(x, t) = \bar{\rho} + \phi(x, t)$ into the mean density and a fluctuating part $\phi(x, t)$, which satisfies the diffusion equation

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2}. \tag{2}$$

Use this to derive the time evolution of the intermediate structure function

$$S(k, t) = \langle \hat{\phi}(k, t) \hat{\phi}(-k, 0) \rangle \tag{3}$$

where brackets refer to the thermal equilibrium state and

$$\hat{\phi}(k, t) = \int dx\, e^{ikx} \phi(x, t). \tag{4}$$

You should find

$$S(k, t) = e^{-|k|^2 D |t|} S(k, 0) = L\bar{\rho}(1 - \bar{\rho}) e^{-|k|^2 D |t|}. \tag{5}$$

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[1] S. Alexander, P. Pincus, Phys. Rev. B **18**, 2011 (1978)

In the continuum limit the equal time density correlation function is $\langle\phi(x,t)\phi(x',t)\rangle = \bar{\rho}(1-\bar{\rho})\delta(x-x')$, and hence $S(k,0) = L\bar{\rho}(1-\bar{\rho})$, where L is the system size.

To compute the mean square displacement $\langle X_0^2\rangle$, use (1) to relate it to the correlation function of the current $j(0,t)$. The latter can then be obtained from the density correlation function (5) using the fact that $j = -D\partial\phi/\partial x$. The calculation is simplified by introducing the *particle counting function*

$$N(x,t) = \int_0^x dy\, \phi(y,t), \tag{6}$$

which satisfies the relations $j(0,t) = \partial N/\partial t$ and $\phi(x,t) = \partial N/\partial x$. Finally, examine how $\langle X_0(t)^2\rangle$ behaves in the low density limit $\bar{\rho} \to 0$, and provide an interpretation.

Problem 9: The zero range process

In the zero range process (ZRP)[2] an unlimited number $n_i = 0, 1, 2, ...$ of particles can occupy each site $i = 1, .., L$ of the one-dimensional lattice with periodic boundary conditions (a ring). A particle at site i jumps to the right ($i \to i+1$) with probability p and to the left ($i \to i-1$) with probability $1-p$ at a rate which is a function $\gamma(n_i)$ of the number of particles at the site of origin with $\gamma(0) = 0$. There is no dependence on the occupancy of the target site (= *zero range interaction*). The ZRP has the remarkable property that the stationary distribution is a *product measure* for a broad class of functions $\gamma(n)$, i.e. the stationary weight of a configuration $\{n_1, ..., n_L\}$ is of the form

$$\mathrm{Prob}[n_1, ..., n_L] \sim \prod_{i=1}^{L} f(n_i). \tag{7}$$

a.) In the symmetric case $p = 1/2$ use the condition of detailed balance to show that

$$f(n) \sim \prod_{k=1}^{n} \gamma(k)^{-1} \tag{8}$$

You may ignore the constraint of constant total particle number $N = \sum_{i=1}^{L} n_i$, i.e. work in a "grand-canonical" setting.

b.) Write down the master equation for the asymmetric case and show that the product measure (7, 8) remains stationary also when $p \neq 1/2$.

c.) Under what conditions on the rate function $\gamma(n)$ is the single-site probability distribution (8) normalizable?

[2] F. Spitzer, Adv. Math. **5**, 246 (1970); M.R. Evans, T. Hanney, J. Phys. A **38**, R195 (2005).