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Problems in Advanced Statistical Physics

Problem 10: Detailed balance and reversibility

A continuous time Markov chain on a finite set of states $i = 1, ..., C$ satisfies detailed balance iff

$$\gamma_{ij} P_i^* = \gamma_{ji} P_j^* \tag{1}$$

where γ_{ij} is the transition matrix and P_i^* the stationary distribution. We will assume in the following that the Markov chain is *ergodic*, in the sense that every state can be reached from every other state through a sequence of transitions with nonzero rates.

It is often necessary to check for the presence or absence of detailed balance in situations where the stationary distribution P_i^* is not explicitly known. To this end consider a closed loop

$$\mathcal{L} = (i_1 \to i_2 \to i_3 \to ... \to i_N \to i_1)$$

in state space along with its time-reversed partner $\bar{\mathcal{L}} = (i_1 \to i_N \to i_{N-1} \to ... \to i_2 \to i_1)$. Take the product of the transition rates along the loop,

$$\pi(\mathcal{L}) \equiv \prod_{k=1}^{N} \gamma_{i_k i_{k+1}}$$

with $i_{N+1} \equiv i_1$, and show that (1) holds if and only if $\pi(\mathcal{L}) = \pi(\bar{\mathcal{L}})$ for *all* possible loops. Note that this requires in particular to construct (at least in principle) the stationary distribution P_i^* from the rates.

Problem 11: Shocks in the viscous Burgers equation

The one-dimensional Burgers equation

$$\frac{\partial \phi}{\partial t} + \lambda \phi \frac{\partial \phi}{\partial x} = \nu \frac{\partial^2 \phi}{\partial x^2} \tag{2}$$

was originally introduced as a model problem for fluid turbulence[1]. In that context $\lambda = 1$ because of Galilean invariance, and $\nu > 0$ is the kinematic viscosity.

The inviscid Burgers equation with $\nu = 0$ describes the behavior of the asymmetric exclusion process in the hydrodynamic limit. It was shown in the lectures that the inviscid equation

[1] J.M. Burgers, *The Nonlinear Diffusion Equation* (Riedel, Boston 1974).

generically develops shock discontinuities from smooth initial conditions. Here we want to investigate how the viscosity regularizes the shocks.

Show that (2) admits stationary traveling wave solutions of the form

$$\phi(x,t) = \Phi(x - Vt), \tag{3}$$

where the shape function is asymptotically constant, $\lim_{\xi \to -\infty} \Phi(\xi) = \phi_L$ and $\lim_{\xi \to \infty} \Phi(\xi) = \phi_R$. Find a first integral of the ordinary differential equation for $\Phi(\xi)$ and use it to determine the velocity V in terms of the boundary values ϕ_L and ϕ_R. Then compute the function $\Phi(\xi)$ explicitly and show that the solution reduces to a discontinuous shock in the limit $\nu \to 0$.

Problem 12: Cellular automaton rule 184

Cellular automata (CA) are dynamical systems with discrete spatial structure evolving in discrete time. Elementary CA in the sense of Wolfram[2] are defined on a one-dimensional lattice with binary variables on each site, and the state of a site at time t is a deterministic (Boolean) function of the state of the site itself and its two neighbors at time $t-1$. There is a total of $2^8 = 256$ such CA which can be completely classified. Here we consider rule 184 defined by

$$(000) \to 0,\ (001) \to 0,\ (010) \to 0,\ (011) \to 1,\ (100) \to 1,\ (101) \to 1,\ (110) \to 0,\ (111) \to 1$$

The sequence of final states is a binary representation of the number $184 = 0 \times 2^0 + 0 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 + 1 \times 2^4 + 1 \times 2^5 + 0 \times 2^6 + 1 \times 2^7$. Rule 184 is a deterministic version of the asymmetric exclusion process with $p = 1$: In one time step all particles with a vacant neighbor site move *simultaneously* to the right. Correspondingly rule 226 describes particles moving deterministically to the left.

a.) Identify all elementary CA that conserve the number of 1's, and show that rules 184 and 226 are the only nontrivial examples.

b.) Determine (by inspection or simulation) the attractor of CA 184 on a finite ring, i.e., the set of configurations that govern the dynamics for $t \to \infty$. Deduce that the stationary particle current is given as a function of density by the expression

$$J(\rho) = \min(\rho, 1 - \rho). \tag{4}$$

The cusp singularity at $\rho = 1/2$ signals a phase transition. What is a suitable order parameter characterizing the transition?

c.) Consider the (inviscid) hydrodynamic equation associated with the current (4). Investigate the behavior of the characteristics. Under what conditions and on which time scale do shocks form from a smooth initial density profile?

[2] S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983); S. Wolfram, *A new kind of science* (2002)