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Problems in Advanced Statistical Physics

Problem 16: Correlations and boundary effects for the Ising chain

a.) Calculate the partition function for the Ising chain in zero magnetic field for free and fixed ($++$ and $+-$) boundary conditions, and show that it reduces to the result for periodic boundary conditions when the system size L becomes large compared to the correlation length.

b.) Derive the spin-spin correlation function for nonzero magnetic field, and verify the relation $\xi^{-1} = \ln(\lambda_1/\lambda_2)$.

Problem 17: Solid-on-solid approximation for the Ising domain wall

Consider a domain wall in the two-dimensional Ising model which runs on average along the x-axis of the lattice. In the solid-on-solid (SOS) approximation configurations with overhangs are neglected, so that the domain wall can be represented by an integer-valued height function $y = h(x)$. For a lattice of linear size L the energy of such a configuration is then

$$H_{\text{SOS}} = 2JL + 2J \sum_{x=1}^{L} |h(x+1) - h(x)|. \tag{1}$$

The mean orientation of the domain wall can be fixed by imposing "helical" boundary conditions $h(L+1) = h(1) + L\tan(\theta)$, where θ is the angle between the domain wall and the x-axis.

The expression (1) can be viewed as the energy of noninteracting slope variables $u(x) = h(x+1) - h(x)$ subject to the global constraint

$$h(L+1) - h(1) = \sum_{x=1}^{L} u(x) = L\tan(\theta), \tag{2}$$

corresponding to a "canonical" ensemble for the "particle numbers" $u(x)$. For actual computations it is convenient to use a grand-canonical ensemble where the energy of a slope configuration is given by

$$H_{\text{S0S}}^{(\text{gc})} = 2JL + 2J \sum_{x=1}^{L} |u(x)| + \mu \sum_{x=1}^{L} u(x) \tag{3}$$

and the mean orientation is determined by the slope chemical potential μ.

Compute the domain wall free energy per unit length $\gamma_{\rm SOS}(\theta, T)$ from (3), and compare to the exact expressions[1]

$$\gamma(0,T) = 2J + k_B T \ln[\tanh(\beta J)], \quad \gamma(\pi/4, T) = \sqrt{2} k_B T \ln[\sinh(2\beta J)] \tag{4}$$

obtained by taking into account *all* configurations. Determine the temperature(s) at which γ and $\gamma_{\rm SOS}$ vanish. What kind of bound on the Ising critical temperature do you expect the SOS-approximation to provide?

Problem 18: Toom's CA

In the two-dimensional Ising model, phase coexistence is possible only along the line $h = 0$ in the (T, h)-plane. In 1980, Andrei Toom devised an Ising-like stochastic cellular automaton in which phase coexistence occurs in an extended two-dimensional region of the analogous parameter plane[2].

The model is defined on the square lattice and a spin $\sigma_{x,y} = \pm 1$ is associated with each site $(x, y) \in \mathbb{Z}^2$. In the deterministic limit (corresponding to temperature $T = 0$) each spin evolves in discrete time by taking on the majority value among itself and its *northern* and *eastern* neighbor, i.e.

$$\sigma_{x,y}(t+1) = \mathrm{sgn}[\sigma_{x,y}(t) + \sigma_{x+1,y} + \sigma_{x,y+1}]. \tag{5}$$

To mimick the effects of temperature and magnetic field, the rule (5) is obeyed only with probability $1 - p - q$; otherwise the spin at time $t+1$ is set equal to 1 (-1) with probability p (q). Thus temperature corresponds roughly to $p + q$ and magnetic field to $p - q$ (for $p = q$ the dynamics is symmetric under $\sigma_{x,y} \to -\sigma_{x,y}$).

To understand the robustness of phase coexistence in this model, investigate (by inspection or simulation!) the time evolution of a droplet (of convex initial shape) of (+)-spins inside a sea of (–)-spins, under the deterministic rule (5). Show that the droplet disappears in a time that is proportional to its linear extent, and show that the same is true for a droplet of (–)-spins inside a sea of (+)-spins. This behavior is not changed by a small amount of noise, and therefore both phases remain stable for generic (small) values of p and q.

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[1] C. Rottman and M. Wortis, Phys. Rev. B *24* 6274 (1981).

[2] A.L. Toom, in *Multicomponent random systems*, ed. by R.L. Dobrushin (New York, 1980); G. Grinstein, IBM J. Res. & Dev. **48** (2004), available on the course web page.