UNIVERSITÄT ZU KÖLN

Prof. Dr. Joachim Krug
Dr. Su-Chan Park
Institut für Theoretische Physik
WS 2007/2008

Problems in Advanced Statistical Physics

Problem 19: Ising chain with long ranged interactions [1]

The Ising chain with a power law interaction is defined by

$$H = -\sum_{i=1}^{L} \sum_{j<i} \frac{J}{|i-j|^\alpha} \sigma_i \sigma_j \tag{1}$$

where periodic boundary conditions are assumed. Following the arguments of Sect.II.2.3 of the lectures, estimate the energy cost associated with a domain wall in this system. For which values of α would long ranged order at finite temperature be possible?

Problem 20: Domain walls in the Ginzburg-Landau theory The Ginzburg-Landau (GL) domain wall is an order parameter profile of the form

$$\phi_{DW}(x) = \phi_0 \tanh(x/w), \tag{2}$$

where $\phi_0 = \sqrt{-a/b}$ is the value of the order parameter in the low temperature phase and $w = \sqrt{-g/2a}$ is the domain wall width, proportional to the correlation length ξ.

a.) Compute the free energy of the domain wall, defined as the difference between the GL functional evaluated for the OP profile (2) and the homogeneous profile $\phi \equiv \phi_0$. Investigate how the domain wall free energy behaves at the critical point, and compare to the results for the two-dimensional Ising domain wall (Problem 17).
Hint: Make use of the law of energy conservation for the mechanical analog of the Euler-Lagrange equation.

b.) The solution (2) was obtained by requiring that the first variation of the GL functional vanishes. To prove that it is actually a minimum, we should check that the second variation $\delta^2\mathcal{F} \geq 0$. Here $\epsilon^2\delta^2\mathcal{F}$ is the contribution quadratic in ϵ to the change $\delta\mathcal{F}$ of the GL free energy, when the profile $\phi(\vec{r})$ is perturbed according to $\phi(\vec{r}) \to \phi(\vec{r}) + \epsilon\psi(\vec{r})$. We restrict ourselves here to one-dimensional perturbations $\psi(x)$ of the profile (2).

[1] M. Plischke & B. Bergersen, *Equilibrium statistical physics*, Problem 3.1.

Show that[2]

$$\delta^2 \mathcal{F} = \langle \psi | \mathcal{H} | \psi \rangle, \tag{3}$$

where $\mathcal{H}$ has the form of a one-dimensional Schrödinger operator with potential $W(x) = (1/2) f''(\phi_{DW}(x))$. Sketch the potential, and show that it has an eigenfunction with zero energy. Argue (using general results from one-dimensional quantum mechanics) that this is the ground state of the potential, and hence that indeed $\delta^2 \mathcal{F} \geq 0$. What is the physical meaning of this eigenfunction, and why does it have zero energy? Can you find other eigenfunctions?

c.) In a small magnetic field $h \neq 0$ the domain wall will move to increase the size of the favored domain. To model this process, we consider the *time-dependent GL equation*

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta \mathcal{F}}{\delta \phi}, \tag{4}$$

where $\Gamma > 0$ is a kinetic coefficient. As before, we restrict to one-dimensional profiles $\phi(x, t)$. Show that, in the presence of a magnetic field, (4) permits a traveling wave solution of the form

$$\phi(x, t) = \Phi(x - Vt). \tag{5}$$

Write down the equation for the profile function $\Phi(z)$, and intepret it in terms of classical mechanics. Show, using the mechanical analogy, that the front velocity V in (5) is uniquely determined by the field, and derive an expression for the mobility $\sigma = \lim_{h \to 0} V/h$ of the front. As in part a.), it is useful to consider the energy balance in the mechanical analog.

[2]See e.g. J.S. Langer, Ann. Phys. **65**, 53 (1971).