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Problems in Advanced Statistical Physics

Problem 21: The Fisher-Kolmogorov equation

The one-dimensional Fisher-Kolmogorov equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \phi(1-\phi) \tag{1}$$

was suggested by R.A. Fisher as a model for the spread of an advantageous gene in a population[1], and subsequently analyzed by Kolmogorov, Petrovsky and Piskunov. It is a paradigm for front propagation with many applications in biology, chemistry and physics[2]. In applications, the field ϕ is usually restricted to $0 \leq \phi \leq 1$.

a.) Stationary homogeneous solutions of (1) are obviously $\phi \equiv 0$ and $\phi \equiv 1$. Show that the solution $\phi = 1$ is *stable* while $\phi = 0$ is *unstable*. To see this, linearize (1) around a homogeneous state ϕ_0, $\phi(x,t) = \phi_0 + \epsilon(x,t)$, and look for solutions of the linearized problem of the form

$$\epsilon(x,t) \sim \exp[iqx + \omega(q,\phi_0)t]. \tag{2}$$

The homogeneous state is unstable if $\omega(q,\phi_0) > 0$ for some q.

b.) Show that (1) can be written in the form of a time-dependent Ginzburg-Landau equation, as introduced in Problem 20, and determine the corresponding Landau free energy $f(\phi)$. What is the thermodynamic interpretation of the instability of the state $\phi = 0$?

c.) Using the mechanical analogy of Problem 20 c.), show that (1) possesses traveling wave solutions

$$\phi(x,t) = \Phi(x - Vt), \tag{3}$$

with $0 \leq \Phi \leq 1$ and boundary conditions $\lim_{z\to\infty} \Phi = 0$, $\lim_{z\to-\infty} \Phi = 1$, for *all* velocities $V \geq V_{\min} > 0$, and determine the minimal speed $V_{\min}$. This is in contrast to the field-driven interfaces in Problem 20 c.), where the speed is uniquely determined.

d.) The degeneracy of the front speed in (3) implies that the actual speed must depend on the initial condition. The full theory of the Fisher-Kolmogorov equation and related systems[3] shows that spatially localized initial conditions typically lead to propagation of the interface at the *linear spreading speed* V^*, which is defined as the speed at which points of constant ϕ propagate under the *linear* dynamics obtained by linearizing (1) around the unstable state $\phi = 0$. Find the general solution of the linear equation, and determine V^*. Show that $V^* = V_{\min}$, i.e. the front selects the *minimal* speed.
Hint: Consider the transformation $\phi(x,t) = e^t u(x,t)$.

[1] R.A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugenics **7**, 353 (1937).
[2] J.D. Murray, *Mathematical Biology* (Springer, 2002).
[3] W. van Saarloos, Phys. Rep. **386**, 29 (2003).

Problem 22: Phase fluctuations in the XY-model

In the lectures the following expression was derived for the phase correlation function of the XY-model in the spin wave (=low temperature) approximation:

$$\langle (\varphi(\vec{r}) - \varphi(0))^2 \rangle = \frac{1}{V} \sum_{\vec{q}} (1 - \cos(\vec{q} \cdot \vec{r})) \frac{k_{\rm B} T}{g |\vec{q}|^2}, \tag{4}$$

where the sum runs over discrete wave vectors $\vec{q}$ consistent with (e.g.) periodic boundary conditions at the boundaries of the d-dimensional domain of volume V.

a.) Evaluate (4) exactly in one dimension. Note that in this case the small scale cutoff Λ can be sent to infinity, and make use of the series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}.$$

b.) In two dimensions evaluate (4) approximately by converting the sum into an integral and computing the latter for the circular domain $|\vec{q}| \leq \Lambda$. Make use of the identity

$$\int_0^{2\pi} d\theta \; \cos(x \cos\theta) = 2\pi J_0(x)$$

and of the properties of the zero'th order Bessel function J_0 to show that indeed (4) diverges as $(k_{\rm B} T / \pi g) \ln(r\Lambda)$ for large r.

Problem 23: Fluid interfaces under gravity

a.) Show that in the presence of gravity, the capillary wave free energy of a fluid interface (say, between a liquid and its vapor) takes on the form

$$\mathcal{F} = \int d^2\vec{x} \left[\frac{1}{2} \gamma (\nabla h)^2 + \frac{1}{2} \alpha h^2 \right] \tag{5}$$

and determine the coefficient α in terms of the gravitational acceleration g and the mass density difference $\Delta\rho$ of the two fluid phases.

b.) Compute the height difference correlation function $C(\vec{x})$ from (5), and show that the logarithmic divergence derived in the lectures saturates for $|\vec{x}| \gg \xi_c$, where $\xi_c = \sqrt{\gamma/\alpha}$ is the *capillary length*. The limiting value

$$W^2 \equiv \lim_{|\vec{x}| \to \infty} C(\vec{x}) \tag{6}$$

is a possible definition for the interfacial width W induced by the capillary waves. Estimate ξ_c and W^2 for water at room temperature ($\gamma = 0.073$N/m, $\Delta\rho = 10^3$kg/m^3).

c.) Investigate the behavior of ξ_c and W on approaching the critical point. Use the critical exponents of Landau theory. In particular, use the result of Problem 20 a.) for the behavior of the interface free energy γ near T_c. Compare the behavior of ξ_c and W with that of the bulk correlation length ξ, which determines the *intrinsic* width of the interface.