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Problems in Advanced Statistical Physics

Problem 24: Real space RG for percolation

Percolation is an example of an athermal, purely geometric phase transition[1]. Consider a d-dimensional finite lattice on which sites are occupied or vacant independently with probability p or $1-p$. Then percolation occurs if a *cluster* consisting of occupied nearest neighbor sites exists which spans the entire system, that is, it reaches from one boundary of the system to another. In the thermodynamic limit the probability of percolation is zero below a well-defined threshold p_c and finite for $p > p_c$. For $p < p_c$ all connected clusters are finite, while for $p > p_c$ the finite clusters coexist with an infinite cluster containing a finite fraction of all sites.

In this problem the real space RG method is applied to percolation on the two-dimensional triangular lattice. As in the case of the Ising model considered in the lectures, the RG transformation maps a triangular plaquette of three sites onto a single site of the 'blocked' lattice, so $b = \sqrt{3}$. The renormalized occupation probability $p' = \mathcal{R}_b(p)$ is the probability that a plaquette contains a spanning cluster of occupied sites, i.e. a cluster that extends from one corner of the plaquette to another.

Show that, out of the $2^3 = 8$ possible configurations of the plaqutte, 4 contain a spanning cluster, and compute p' as a function of p. Show that the RG transformation $\mathcal{R}_b(p)$ has attractive fixed points at $p = 0$ and $p = 1$, and a repulsive critical fixed point[2] at a value $0 < p_c < 1$. Compute the correlation length exponent ν from the relation $b^{1/\nu} = d\mathcal{R}_b/dp|_{p=p_c}$ and compare to the exact value $\nu = 4/3$.

Problem 25: Stable laws and renormalization [3]

Consider two independent real random variables X_1 and X_2 drawn from the same distribution $P(X)$, which is assumed for simplicity to have zero mean. The distribution $P_\Sigma(Y)$ of the sum $Y = X_1 + X_2$ is then given by

$$P_\Sigma(Y) = \int_{-\infty}^{\infty} dX \; P(Y - X)P(X). \tag{1}$$

[1] But it is also equivalent to the thermal Potts model in the limit $q \to 1$.

[2] The exact percolation threshold for the triangular lattice is at $p_c = 1/2$.

[3] See Excercise 12.1. in the book of J.P. Sethna, and Sect. 2.3 of *Critical Phenomena in the Natural Sciences* by D. Sornette (Springer, 2000).

A probability distribution P is called a *stable law* if P_Σ is identical to P under appropriate rescaling of the argument, that is if

$$\mathcal{R}_b[P](Y) \equiv bP_\Sigma(bY) = b\int_{-\infty}^{\infty} dX\ P(bY - X)P(X) = P(Y) \tag{2}$$

for a suitably chosen scale factor b. Obviously (2) can be viewed as a fixed point condition for a kind of renormalization transformation $\mathcal{R}_b$ acting on the space of probability distributions.

a.) Rewrite the fixed point condition (2) using the generating function

$$G(k) = \int_{-\infty}^{\infty} dX\ e^{ikX} P(x). \tag{3}$$

Show that the Gaussian and the Cauchy distributions

$$P_G(X) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2}, \quad P_C(X) = \frac{1}{X^2 + \pi^2} \tag{4}$$

are fixed points of (2), and identify the corresponding values of b. Show more generally that (2) is solved by the one-parameter family of (symmetric) Lévy stable laws with generating function $e^{-C|k|^{\mu}|}$, where $0 < \mu \leq 2$.

b.) The central limit theorem is a statement about the basin of attraction of the Gaussian fixed point of (2). To investigate the stability of this fixed point, linearize $\mathcal{R}_b$ around P_G. Show that the eigenfunctions of the corresponding linear operator are of the form $k^n e^{-k^2/2}$ in Fourier space, and determine the eigenvalues. What is the significance of the non-negative eigenvalues associated with relevant and marginal perturbations?

Problem 26: Power counting for the self-avoiding walk

The Edwards Hamiltonian[4]

$$\mathcal{F}_{\mathrm{E}} = \int ds\, \frac{1}{2}\left(\frac{d\vec{r}}{ds}\right)^2 + u\int ds \int ds' \delta^d(\vec{r}(s) - \vec{r}(s')) \tag{5}$$

is the starting point for a continuum field theory description of self-avoiding random walks. Here $\vec{r}(s)$ is the conformation of the polymer in d-dimensional space, parametrized by the arc length s. For $u = 0$ (5) is the weight of a sample path in the Wiener process, and for $u > 0$ the second term takes into account the energetic cost of self-intersections.

a.) Analyze the relevance of u at the Gaussian fixed point $u = 0$ by power counting, and show that the upper critical dimension for self-avoidance is $d_> = 4$.

b.) Now assume that $\vec{r}(s)$ transforms under rescaling of s, $s \to s' = bs$, as $\vec{r} \to b^\nu \vec{r}$, where ν can differ from the value $\nu = 1/2$ associated with the Gaussian fixed point. Determine ν from the requirement that the two terms in (5), with u kept fixed, should tranform in the same way under rescaling. What do you find?

[4] S.F. Edwards, Proc. Phys. Soc. London **88**, 265 (1966).