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Solutions in Advanced Statistical Physics

Solution 19. Ising chain with long-ranged interactions

Let us check the extensivity of the Hamiltonian. The ground state energy is

$$E_0 = -\frac{1}{2}\sum_{i=1}^{L}\sum_{j\neq i}\frac{J}{|j-i|^\alpha} \approx -L\int_1^{L/2}\frac{J}{y^\alpha}dy \propto -LJ(L^{1-\alpha}+const),$$

where absolute value $|x|$ should be understood as $L-|x|$ if $|x|>L/2$ (periodic boundary conditions) and the sum is approximated by an integral. Hence the energy is extensive when $\alpha>1$. Otherwise, that is, when $\alpha\le 1$, we have to rescale the coupling constant as $J\mapsto \frac{J}{L^{1-\alpha}}$ to make the energy extensive. Note that $\alpha=0$ corresponds to the Ising model on a complete graph (every spin interacts with every other spin with equal strength) which is solvable exactly and shows a mean-field phase transition at finite temperature.

Now go back to the domain wall arguments. The energy cost associated with a domain of n flipped spins is ($L\gg n$)

$$\Delta E_n = 2\sum_{i=1}^{n}\sum_{j=n+1}^{L}\frac{J}{|j-i|^\alpha} \approx 4\sum_{i=1}^{n}\sum_{j=n+1}^{L/2}\frac{J}{|j-i|^\alpha}.$$

Approximating the sum by an integral, we get

$$\Delta E_n \simeq 4J\int_1^n dx\int_{n+1}^{L/2}\frac{dy}{|y-x|^\alpha} \approx \frac{4J}{1-\alpha}\left[\left(\frac{L}{2}\right)^{1-\alpha}n - \frac{n^{2-\alpha}}{2-\alpha}\right] \propto \begin{cases} n & \text{if } \alpha\le 1\\ n^{2-\alpha} & \text{if } \alpha>1\end{cases} \tag{1}$$

where we assumed $L\gg n\gg 1$ and used the proper rescaling as explained above. To have a finite critical temperature, the energy cost of the domain should increase indefinitely with n. In this case, at low temperature the system cannot afford to make a big domain by thermal fluctuations. This corresponds to the condition $\alpha<2$.

Solution 20. Domain walls in the Ginzburg-Landau theory

a.) The free energy of the domain wall (DW) is defined as $F_{DW} = F[\phi_{DW}(x)] - F[\phi_0]$, or

$$F_{DW} = \int dx \left[\frac{g}{2}\left(\phi'_{DW}\right)^2 + f(\phi_{DW}) - f(\phi_0)\right].$$

From the mechanical analog we have $\frac{g}{2}\left(\phi'_{DW}\right)^2 - f(\phi_{DW}) = -f(\phi_0)$, hence

$$F_{DW} = g\int \left(\phi'_{DW}\right)^2 dx = g\frac{\phi_0^2}{w^2}\int \frac{dx}{\cosh^4(x/w)} = \left(\frac{8g|a|^3}{9b^2}\right)^{1/2},$$

where $w = \sqrt{-2g/a}$ (note the typo in the problem). As $T \to T_c - 0$, $F_{DW} \sim |a|^{3/2} \sim |T - T_c|^{3/2}$. In contrast, for the Ising DW (problem 17), we had

$$\gamma_{SOS} = 2J + kT\ln\tanh(\beta J) \sim |T - T_c|$$

close to the critical point.

b.) Expanding the Ginzburg-Landau free energy functional up to second order, one obtains

$$F[\phi_{DW} + \epsilon\psi] = F[\phi_{DW}] + \epsilon^2\int dx\left\{-\frac{g}{2}\psi\partial_x^2\psi + \frac{1}{2}f''(\phi)\psi^2\right\},$$

where ϕ_{DW} is the solution of the Euler-Lagrange equation

$$-g\partial_x^2\phi_{DW} + f'(\phi_{DW}) = 0. \tag{2}$$

Hence

$$\delta^2 F = \int dx\left\{-\frac{g}{2}\psi\partial_x^2\psi + \frac{1}{2}f''(\phi_{DW})\psi^2\right\} = \langle\psi|\mathcal{H}|\psi\rangle,$$

where

$$\mathcal{H} = -\frac{g}{2}\partial_x^2 + \frac{1}{2}f''(\phi_{DW}) = -\frac{g}{2}\partial_x^2 + \frac{|a|}{2}\left(3\tanh^2(x/w) - 1\right).$$

Let the eigenvalue of $\mathcal{H}$ be denoted by λ. From Eq. (2), one can deduce that

$$0 = \partial_x(-g\partial_x^2\phi_{DW} + f'(\phi_{DW})) = -g\partial_x^2(\partial_x\phi_{DW}) + f''(\phi_{DW})(\partial_x\phi_{DW}) = 2\mathcal{H}\left|\partial_x\phi_{DW}\right\rangle,$$

which implies $\psi_0 \equiv \partial_x\phi_{DW}$ is an eigenstate of $\mathcal{H}$ with eigenvalue $\lambda = 0$. Since $\psi_0 \propto \mathrm{sech}^2(x/w)$ has no node, this should be the ground state of $\mathcal{H}$. Hence $\langle\psi|\mathcal{H}|\psi\rangle \geq \lambda_{\min}\langle\psi|\psi\rangle = 0$ indeed yields $\delta^2 F \geq 0$[1].

The eigenfunction ψ_0 corresponds to the translational mode of the domain wall solution $\phi_{DW}(x)$: indeed a shift of the solution does not change the free energy of the domain wall. Since the infinitesimal translation by ϵ is described by the operator $\epsilon\frac{\partial}{\partial x}$, we have

$$\delta F[\phi(x) + \epsilon\frac{\partial}{\partial x}\phi(x)] = 0$$

from which, again, $\mathcal{H}\left|\frac{\partial}{\partial x}\phi(x)\right\rangle = 0$ follows to first order in ϵ.

[1]The *oscillation theorem* states that the n-th largest eigenvalue of the discrete spectrum for a Schrödinger operator in one dimension has n nodes for finite x.

- **Other discrete spectrum eigenvalues:** Consider the eigenvalue problem

$$\mathcal{H}\psi = -\frac{g}{2}\psi'' + \frac{|a|}{2}\left(2 - 3\frac{1}{\cosh^2(x/w)}\right)\psi = \lambda\psi.$$

Let $y = x/w$ and $\psi(x) = \varphi(y)$. Then the Schrödinger equation becomes

$$\varphi'' + \frac{6}{\cosh^2(y)}\varphi = -4\left(\frac{\lambda}{|a|} - 1\right)\varphi \equiv \xi\varphi.$$

This turns out to be solvable exactly[2]. The eigenvalues are

$$\xi_n = (2-n)^2 \to \lambda_n = |a|\left[1 - \left(1 - \frac{n}{2}\right)^2\right],\ 0 \le n \le 2$$

and the corresponding ground state is

$$\varphi_0 \sim \frac{1}{\cosh^2(y)},$$

and the excited state is

$$\varphi_1 \sim \frac{\tanh(y)}{\cosh(y)}.$$

c.) The time-dependent GL equation with the traveling wave ansatz becomes

$$-V\Phi' = -\Gamma\left(-g\Phi'' + \frac{\partial f}{\partial \Phi}\right). \tag{3}$$

where the Landau free energy in this problem is $f(\Phi) = -h\Phi + a\Phi^2/2 + b\Phi^4/4$ $(h > 0)$. Equation (3) can be written as a particle in a potential $U(\Phi) = -f(\Phi)$ with a dissipative term (the spatial coordinate plays the role of 'time' and Φ is the 'spatial coordinate' of the 'particle'):

$$g\Phi'' = \frac{\partial f}{\partial \Phi} - \frac{V}{\Gamma}\Phi'.$$

The traveling wave is interpreted as the particle trajectory starting at the position of the larger maximum of the potential Φ_R at infinite past $x = -\infty$ and arriving at the lower potential maximum Φ_L at $x = \infty$ ($\Phi(x)$ takes the form of $-\tanh(x)$). The energetic balance is obtained, multiplying the equation of motion by Φ' and integrating over the whole 'time' axis,

$$\frac{V}{\Gamma}\int (\Phi')^2\, dx = f(\Phi_L) - f(\Phi_R) \tag{4}$$

(the total energy dissipation is equal to the potential difference between initial and end point). Equations (3) and (4) fully specify the traveling wave solution.

The uniqueness of V, if the existence is guaranteed, can be understood along the lines of the mechanical analog. Let the solution of (4) be V_0. If $V > V_0$, the trajectory

[2]See, e.g., Landau course on theor. physics; volume 3 (Nonrelativistic quantum mechanics).

cannot end at the other maximum. Rather, the strong damping keeps the particle from arriving at Φ_L and the 'particle' eventually will stop at the local minimum of U. On the other hand, weak damping will drive the particle to escape the potential barrier. Hence V_0 is determined uniquely.

Let us find the approximate solution of V for $h \ll 1$. First observe that

$$\Phi_R \approx \phi_0 + \frac{1}{2a}h, \quad \Phi_L \approx -\phi_0 + \frac{1}{2a}h, \quad f(\Phi_L) - f(\Phi_R) \approx 2h\phi_0.$$

Since $\Phi \to -\phi_{DW}(x)$ as given in a.) and $V \to 0$ as $h \to 0$, Eq. (4) to leading order in h becomes

$$\frac{V}{\Gamma}\int (\Phi'(x))^2 \approx \frac{V}{\Gamma g}F_{DW} \approx 2h\phi_0. \tag{5}$$

Hence the mobility of the domain wall is

$$\sigma = \lim_{h\to 0}\frac{V}{h} = \frac{2\phi_0 \Gamma g}{F_{DW}} = \frac{3\Gamma}{-a}\sqrt{\frac{gb}{2}}.$$

Due to the symmetry under the transformation $h \mapsto -h$ and $\Phi \mapsto -\Phi$, the mobility does not change if $h < 0$.