UNIVERSITÄT ZU KÖLN

Prof. Dr. Joachim Krug
Dr. Su-Chan Park
Institut für Theoretische Physik
WS 2007/2008

Solutions in Advanced Statistical Physics

Solution 21. The Fisher-Kolmogorov equation

a.) linear stability analysis

$$\partial_t \epsilon \simeq \partial_x^2 \epsilon + (1 - 2\phi_0)\epsilon, \Rightarrow \omega(q, \phi_0) = -q^2 + (1 - 2\phi_0). \tag{1}$$

For $\phi_0 = 1$, ω is negative for all q's while ω becomes positive for $\phi_0 = 0$ when $q < 1$. Hence $\phi_0 = 1(0)$ is the stable (unstable) state.

b.) analogy to the time-dependent Ginzburg-Landau equation

Let $f(\phi) = -\phi^2/2 + \phi^3/3$. Then the Fisher-Kolmogorov equation becomes the time-dependent Ginzburg-Landau equation (problem 20) with $\Gamma = 1$ and the functional

$$\mathcal{F} = \int dx \left(\frac{1}{2} (\nabla \phi)^2 + f(\phi) \right). \tag{2}$$

$\phi = 0$ is the (local) maximum of the Landau free energy which is unstable.

c.) traveling wave solution

$$\partial_\xi^2 \Phi + V \partial_\xi \Phi + \Phi(1 - \Phi) = 0, \tag{3}$$

where $\xi = x - Vt$. Equation (3) can be interpreted as the damped motion of a particle with unit mass under the potential $U(\Phi) = \Phi^2/2 - \Phi^3/3$ which has minimum at $\Phi = 0$. When damping constant V is large enough, the particle cannot overshoot the point $\Phi = 0$, which is required because of the condition $0 \le \Phi \le 1$. Hence the motion of the particle close to ϕ_0 should behave exponentially $\sim e^{-\kappa t}$. From Eq. (3), we obtain

$$\kappa^2 - V\kappa + 1 = 0, \tag{4}$$

which has real solutions if $V \ge 2 = V_{\min}$.

d.) selection of the front speed

The linear equation around the unstable state $\phi = 0$ is

$$\partial_t \phi = \partial_x^2 \phi + \phi \qquad [\text{put } \phi = e^t u(x,t)] \to \partial_t u(x,t) = \partial_x^2 u(x,t). \tag{5}$$

To solve the partial differential equation, let $u(x,t) = \frac{1}{2\pi}\int dq e^{iqx}\tilde{u}(q,t)$ with $\tilde{u}(q,0) = 1$ which corresponds to the initial condition $u(x,0) = \delta(x)$. Hence

$$\partial_t \tilde{u}(q,t) = -q^2\tilde{u}(q,t) \to \tilde{u}(q,t) = e^{-q^2 t} \Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}.$$

From $\phi(x,t) = e^t u(x,t) = C$ (constant), one can get

$$x_C(t) = 2t\left(1 - t^{-1}\ln\left(C\sqrt{4\pi t}\right)\right)^{1/2} \Rightarrow V^* = \lim_{t\to\infty}\frac{x_C(t)}{t} = 2, \tag{6}$$

which is $V_{\min}$ in c.).

Solution 22. Phase fluctuations in the XY-model

a.) Let $q = 2\pi n/L$ ($n = 1, 2, \ldots, L$) and $x = 2\pi r/L$. The expression for the phase difference correlation function derived in the lectures is[1]

$$\begin{aligned}\langle(\varphi(r) - \varphi(0))^2\rangle &= \frac{1}{V}\sum_{\vec{q}\neq 0} 2\left(1 - \cos(\vec{q}\cdot\vec{r})\right)\frac{k_B T}{g|\vec{q}|^2} \\ &= \frac{k_B T L}{2\pi^2 g}\sum_{n=-\infty}^{\infty}\frac{1-\cos(nx)}{n^2} \\ &= \frac{k_B T L}{\pi^2 g}\left(\frac{\pi x}{2} - \frac{x^2}{4}\right) = \frac{k_B T}{g} r + O(1/L),\end{aligned}$$

where $n = 0$ is excluded in the sum over n.

b.) Using $\frac{1}{V}\sum_{\vec{q}} \mapsto \frac{1}{(2\pi)^2}\int d^2 q$, the summation becomes

$$\frac{k_B T}{2g\pi^2}\int q dq d\theta \frac{(1 - \cos(qr\cos\theta))}{q^2} = \frac{k_B T}{g\pi}\int_0^{\Lambda} dq \frac{1 - J_0(qr)}{q} = \frac{k_B T}{g\pi}\int_0^{r\Lambda} dx \frac{1 - J_0(x)}{x}.$$

Since

$$J_0(x) = \begin{cases} 1 - \frac{x^2}{4} + O(x^4) & \text{for } x \ll 1, \\ \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) & \text{for } x \gg 1, \end{cases}$$

the integral is dominated by $\int dx/x \sim \ln(r\Lambda)$.

Solution 23. Fluid interfaces under gravity

a.) free energy in the presence of the gravity

The gravitational energy at $\vec{x}$ is $\int d\vec{x} dz \rho(z) g z$ with $\rho(z) = \rho_0(\rho_1)$ if $z < (>) h(\vec{x})$. Hence the energy at $\vec{x}$ becomes

$$\int_0^L dz \rho(z) z g = \frac{1}{2}\Delta\rho g h^2 + \text{ constant}, \tag{7}$$

[1] Note that in the problem set a factor 2 was missing in the first relation.

where $\Delta\rho = \rho_0 - \rho_1$. The free energy takes the form as in the problem sheet with $\alpha = \Delta\rho g$.

b.) the capillary length and the interfacial width

From the lecture,

$$C(r) \simeq \frac{2k_BT}{(2\pi)^2}\int_0^\Lambda d^2q \frac{1-\cos(qr)}{\gamma q^2+\alpha} = \frac{2k_BT}{(2\pi)^2\gamma}\int_0^{\xi_c\Lambda} d^2q \frac{1-\cos(qx)}{q^2+1}, \tag{8}$$

where $\xi_c = \sqrt{\gamma/\alpha}$ and $x = r/\xi_c$. When $x \gg 1$, the main contribution comes from the momentum range $qx \gg 1$. Thus

$$C(r) \simeq \frac{k_BT}{2\pi\gamma}\int_{1/x}^{\xi_c\Lambda} dq \frac{2q}{q^2+1} = \frac{k_BT}{2\pi\gamma}\ln\frac{(\xi_c\Lambda)^2+1}{1+x^{-2}} \xrightarrow{x\to\infty} \frac{k_BT}{2\pi\gamma}\ln((\xi_c\Lambda)^2+1) = W^2. \tag{9}$$

At room temperature, $\xi_c = \sqrt{\gamma/\alpha} = \sqrt{0.073/(10^3 \times 9.8)}\text{m} = 2.7\text{mm}$ and $W \sim 7.8 \times 10^{-10}\text{m} \simeq 8\text{Å}$, where $\Lambda \simeq 1\text{Å}^{-1}$ is used.

c.) near criticality

Since $\gamma \sim t^{3/2}$ (problem 19) and $\alpha \propto \Delta\rho \sim t^{1/2}$ near criticality (t is the reduced temperature), $\xi_c \sim t^{1/2}$ and $W \sim 1/\sqrt{\gamma} \sim t^{-3/4}$. The bulk correlation length diverges as $\sim t^{-1/2}$ which is smaller than W. Actually, W and the bulk correlation length should scale equally, which just means that the mean field theory is wrong in three dimensions.