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Solutions in Advanced Statistical Physics

Solution 24. Real space RG for percolation

The probability that a plaquette contains a spanning cluster is

$$p' = 3p^2(1-p) + p^3 = R_b\,(p)$$

(three configurations of two filled + 1 empty site, one configuration with all three sites of the plaquette filled). Fixed points are given by $R_b\,(p^*) = p^*$ and we readily find $p^* = 0, 1, 1/2$. For the first point $p^* = 0$, and $R_b\,(p^* + \varepsilon) < R_b\,(p^*)$, which corresponds to an attractive fixed point. Analogously $p^* = 1$ is also attractive. $p^* = \frac{1}{2}$ is repulsive, because

$$\partial R_b\,(p)\,/\partial p|_{p=1/2} = \frac{3}{2} > 0.$$

The correlation length exponent is defined by $b^{1/\nu} = \partial R_b\,(p)\,/\partial p|_{p=p_c=1/2} = 3/2$. Substituting $b = \sqrt{3}$ we obtain $\nu \approx 1.355$, which is a very good appoximation to the exact value $\nu_{exact} = 4/3 = 1.3333$. Note also that RG gives the exact critical point.

Solution 25. Stable laws and renormalization

a.) Generating function and stable distributions

Since the generating function of the convolution is

$$\int e^{iky} b P_\Sigma(by) dy = \int dx P(x) e^{i(k/b)x} \int dy b e^{i(k/b)(by-x)} P(by-x) = G(k/b)^2, \qquad (1)$$

the stability condition becomes

$$G(k) = G(k/b)^2. \qquad (2)$$

Since the generating functions of the Gaussian and the Cauchy distributions are

$$G_G(k) = e^{-k^2/2}, \quad G_C(k) = e^{-\pi|k|}, \qquad (3)$$

respectively, the Gauss (Cauchy) distribution is shown to be stable if $b = \sqrt{2}$ (1) is used. In general, the Lévy distribution with the generating function $e^{-C|k|^\mu}$ is stable, if $b = 2^{1/\mu}$ is used.

b.) central limit theorem.

Let $P(x) = P_G(x) + \epsilon f(x)$ with the generating function $G(k) = G_G(k) + \epsilon F(k)$. Then the generating function of the transformed distribution becomes [see Eq. (1)] $G_G(k/\sqrt{2})^2 + 2\epsilon G_G(k/\sqrt{2})F(k/\sqrt{2})$. The eigenvalue problem for the derivative of $\mathcal{R}_b$ at the stable distribution becomes

$$\lim_{\epsilon \to 0} \frac{\mathcal{R}_{\sqrt{2}}[P_G + \epsilon f] - \mathcal{R}_{\sqrt{2}}[P_G]}{\epsilon} = 2G_G(k/\sqrt{2})F(k/\sqrt{2}) = \lambda_n F(k). \tag{4}$$

Since $G_G\left(k/\sqrt{2}\right)^2 = G_G(k)$, Eq. (4) can be rewritten as

$$H\left(k/\sqrt{2}\right) = \frac{\lambda_n}{2} H(k) \tag{5}$$

with $H(k) = F(k)/G_G(k)$. Since $H(k)$ is similar to a homogenous function, we expect $H(k)$ to be a polynomial k^n. Hence the eigenfunction takes the form $F(k) = C(ik)^n G_G(k) = C(ik)^n e^{-k^2/2}$ with the eigenvalue $\lambda_n = 2^{1-(n/2)}$. One can easily see that the inverse Fourier transform of $F(k)$ is

$$f(x) = C\left(-\frac{d}{dx}\right)^n P_G(x), \tag{6}$$

where C is real. Since we are dealing with a probability, the normalization condition requires that $G(k=0) = 1$, in turn $F(k=0) = 0$. Hence $n \leq 0$ is meaningless because of the normalization. Hence we have one relevant ($n = 1$) and one marginal ($n = 2$) perturbation. One can easily check that $f_1(x)$ shifts the mean value in the lowest order of ϵ (hence relevant) and $f_2(x)$ shifts the variance in the lowest order of ϵ (hence marginal because the stability depends on the sign of C).

Solution 26. Power counting for the self-avoiding walk

a.) Power counting and upper critical dimension

The dimension of the conformation $\vec{r}(s)$ is determined from the Wiener measure, requiring $\mathcal{F}_{\mathrm{E}}$ is dimensionless;

$$2\,[\vec{r}] - [s] = 0, \rightarrow [\vec{r}] = -\frac{1}{2}, \tag{7}$$

where $[\ldots]$ stands for the dimension in the momentum scale (inverse of length). By the same token, the dimension of the interaction u is determined;

$$[u] + 2[s] - d[\vec{r}] = 0 \rightarrow [u] = 2 - d/2. \tag{8}$$

Since $[u]$ becomes dimensionless at $d = 4$, the upper critical dimension of the SAW is $d_> = 4$.

b.) rescaling and ν

By rescaling, the Wiener measure becomes

$$(\text{Wiener}) \to b^{1-2\nu}(\text{Wiener}) \tag{9}$$

and the interaction becomes

$$(\text{interaction}) \to b^{-2+d\nu}(\text{interaction}), \tag{10}$$

with u kept fixed. If both terms in the above scale equally, we find

$$1 - 2\nu = -2 + d\nu \to \nu = \frac{3}{2+d}, \tag{11}$$

which is just the prediction of Flory theory.

Solution 27. The Glauber-Ising chain

a.) transition rate

Due to the up-down symmetry, it is sufficient to consider the four local transitions among eight processes:

$$\uparrow\uparrow\uparrow\to\uparrow\downarrow\uparrow\colon \frac{1-\gamma}{2}, \quad \uparrow\downarrow\uparrow\to\uparrow\uparrow\uparrow\colon \frac{1+\gamma}{2}, \quad \uparrow\downarrow\downarrow\leftrightarrow\uparrow\uparrow\downarrow\colon \frac{1}{2}. \tag{12}$$

Detailed balance requires that

$$e^{2K}\frac{1-\gamma}{2} = e^{-2K}\frac{1+\gamma}{2} \to \gamma = \tanh 2K, \tag{13}$$

where $K = \beta J$.

b.) equations for the spin-spin correlation functions

The master equation for the Glauber-Ising chain is

$$\frac{\partial}{\partial t}P(\{\sigma\};t) = \sum_i w_i(-\sigma_i)P(\{\sigma\}'_i;t) - \left[\sum_i w_i(\sigma_i)\right]P(\{\sigma\};t), \tag{14}$$

where $\{\sigma\} \equiv \{\sigma_1, \ldots, \sigma_L\}$, $\{\sigma\}'_i = \{\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_L\}$, and $w_i(\sigma_i) = \Gamma(\sigma_i \to -\sigma_i)$. It is convenient to introduce the state vector $|\Psi;t\rangle$ such that

$$|\Psi;t\rangle = \sum_{\{\sigma\}} P(\{\sigma\};t)|\{\sigma\}\rangle, \tag{15}$$

which, in turn, rewrites the master equation as

$$\begin{aligned}
\frac{\partial}{\partial t}|\Psi;t\rangle &= \sum_{\{\sigma\}}\left\{\sum_i w_i(-\sigma_i)P(\{\sigma\}'_i;t) - \left[\sum_i w_i(\sigma_i)\right]P(\{\sigma\};t)\right\}|\{\sigma\}\rangle \\
&= \sum_{\{\sigma\}}\sum_i w_i(\sigma_i)P(\{\sigma\};t)(\hat{\sigma}_i^x - 1)|\{\sigma\}\rangle \\
&= -\left\{\sum_i (1-\hat{\sigma}_i^x)w_i(\hat{\sigma}_i^z)\right\}|\Psi;t\rangle \equiv -\hat{H}|\Psi;t\rangle,
\end{aligned} \tag{16}$$

where $\hat{\sigma}_i^x$ and $\hat{\sigma}_i^z$ are Pauli matrices which affect only the spin at site i. Introducing the projection state, $\langle\cdot| = \sum_{\{\sigma\}}\langle\{\sigma\}|$, the average of observables can be written as

$$\langle A(\{\sigma\})\rangle = \langle\cdot|A(\{\hat{\sigma}^z\})|\Psi; t\rangle. \tag{17}$$

Using the identity $\langle\cdot|\hat{H} = 0$ which comes from the probability conservation the equation of the average becomes

$$\frac{\partial}{\partial t}\langle A(\{\sigma\})\rangle = \langle\cdot|[\hat{H}, A(\{\hat{\sigma}^z\})]|\Psi; t\rangle. \tag{18}$$

Hence, if $i \neq j$,

$$\begin{aligned}
\frac{dG_{ij}}{dt} &= \left\langle [\hat{\sigma}_i^z, \hat{\sigma}_i^x]\hat{\sigma}_j^z w_i(\hat{\sigma}_i^z) + [\hat{\sigma}_j^z, \hat{\sigma}_j^x]\hat{\sigma}_i^z w_j(\hat{\sigma}_j^z)\right\rangle \\
&= -\left\langle \hat{\sigma}_j^z\left(\hat{\sigma}_i^z - \frac{\gamma}{2}(\hat{\sigma}_{i+1}^z + \hat{\sigma}_{i-1}^z)\right) + \hat{\sigma}_i^z\left(\hat{\sigma}_j^z - \frac{\gamma}{2}(\hat{\sigma}_{j+1}^z + \hat{\sigma}_{j-1}^z)\right)\right\rangle \\
&= -2G_{ij} + \frac{\gamma}{2}\left[G_{i+1,j} + G_{i-1,j} + G_{i,j+1} + G_{i,j-1}\right],
\end{aligned} \tag{19}$$

where we are using $\langle\cdot|\hat{\sigma}_i^x = \langle\cdot|$, $\langle\cdot|\hat{\sigma}_i^z\hat{\sigma}_i^x = -\langle\cdot|\hat{\sigma}_i^z$, and $\hat{\sigma}_i^{z2} = 1$.

c.) stationarity

$$2G_r = \gamma(G_{r+1} + G_{r-1}), \tag{20}$$

with $G_0 = G_L = 1$. Let the solution of the equation $\gamma x^2 - 2x + \gamma x = 0$ be α and β ($\beta > \alpha$).

$$G_r = \frac{1}{\beta - \alpha}\left(\beta^r(G_1 - \alpha) - \alpha^r(G_1 - \beta)\right), \tag{21}$$

where G_1 should be determined from the other boundary condition $G_L = 1$. $\rightarrow G_1 = (\beta - \alpha - \alpha^L\beta + \beta^L\alpha)/(\beta^L - \alpha^L)$

$$G_r = \frac{1}{\beta^L - \alpha^L}(\beta^r(1 - \alpha^L) + \alpha^r(\beta^L - 1)) \xrightarrow[L\to\infty]{} \alpha^r = (\tanh K)^r \tag{22}$$

Easy but non-rigorous way. Expecting $G_r \sim \eta^r$ for large r:

$$\gamma\eta^2 - 2\eta + \gamma = 0 \rightarrow \eta_\pm = \gamma^{-1}(1 \pm (1 - \gamma^2)^{1/2}). \tag{23}$$

Since $\eta_+ > 1$, physical solution should be $\eta_- = \tanh K$.

d.) domain growth at zero temperature

At $T = 0$, $\gamma = 1$.

$$\frac{dG_r}{dt} = -2G_r + G_{r+1} + G_{r-1} \rightarrow \frac{\partial}{\partial t}G(r,t) = \nabla^2 G(r,t), \tag{24}$$

with $G(0,t) = 1$. Putting $G(r,t) = \mathcal{G}(r/t^n)$ into the above equation yields

$$-n\frac{\xi}{t}\mathcal{G}' = \frac{1}{t^{2n}}\mathcal{G}'', \tag{25}$$

where $\xi = r/t^n$. Since $\mathcal{G}$ is a function of the scaling variable ξ only, n should be $\frac{1}{2}$. Hence the equation becomes $\mathcal{G}'' = -\frac{\xi}{2}\mathcal{G}'$ with the constraint $\mathcal{G}(0) = 1$ and $\mathcal{G}(\infty) = 0$. Hence

$$\mathcal{G}(\xi) = C_1 + \tilde{C}_2 \int_0^{\xi} e^{-x^2/4} dx = C_1 + C_2 \int_0^{\xi/2} e^{-x^2} dx, \tag{26}$$

where C_1 and C_2 are constants. From $\mathcal{G}(0) = 1$, $C_1 = 1$ and $\mathcal{G}(\infty)$ give $C_2 = -2/\sqrt{\pi}$. Imposing the obvious symmetry under $r \to -r$, we get $G(r,t) = \mathrm{erfc}(|r|/2\sqrt{t})$, where erfc is the complementary error function. One may be confused by the fact that the solution has no initial condition dependence. The reason should be sought from the continuum limit. Since the continuum limit takes the lattice spacing zero, all initial conditions with finite correlation length on the discrete lattice are reduced to the one given by the solution. For concreteness, consider the correlation function in solution c.) $(\tanh K)^{ar/a}$ with arbitrary K (a lattice spacing). It becomes zero when we take $a \to 0$ limit with ar fixed, except the special point $r = 0$. Hence the continuum limit in a sense selects an initial condition.

When $r/t^n \ll 1$, $\mathcal{G}(\xi) \sim 1 - \xi/\sqrt{\pi} = 1 - r/\sqrt{\pi t}$ which is consistent with Porod's law.

e.) autocorrelation function

Since the solution of the diffusion equation

$$\partial_t P = \frac{1}{2}\nabla^2 P \tag{27}$$

with the delta function initial condition $\delta(r - r')$ at $t = t'$ is

$$P(r - r', t - t') = \frac{1}{\sqrt{2\pi(t - t')}} \exp\left[-\frac{(r - r')^2}{2(t - t')}\right], \tag{28}$$

the two-time correlation function becomes

$$\begin{aligned} G^{(2)}(r,t,t') &= \int \frac{dr'}{\sqrt{2\pi(t-t')}} \exp\left[-\frac{(r-r')^2}{2(t-t')}\right] \mathrm{erfc}(|r'|/2\sqrt{t'}) \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \exp(-(x_0 - x)^2) \mathrm{erfc}(|x|/B), \end{aligned} \tag{29}$$

where $x_0 = r/\sqrt{2(t-t')}$ and $B = \sqrt{2t'/(t-t')}$. Hence the autocorrelation function is

$$\begin{aligned} A(t,t') &= \frac{2}{\sqrt{\pi}} \int_0^{\infty} dx \exp(-x^2) \mathrm{erfc}(x/B) = 1 - \frac{4}{\pi} \int_0^{\infty} dx \int_0^{x/B} dy e^{-x^2-y^2} \\ &= 1 - \frac{4}{\pi} \int_0^{\phi} d\theta \int_0^{\infty} r dr e^{-r^2} = 1 - \frac{2}{\pi}\phi, \end{aligned} \tag{30}$$

where we change the coordinate system into the polar coordinate in 2 dimensions and $\tan\phi = 1/B$. Hence,

$$A(t,t') = \frac{2}{\pi}\left(\frac{\pi}{2} - \arctan(1/B)\right) = \frac{2}{\pi}\arctan B = \frac{2}{\pi}\arctan\sqrt{\frac{2t'}{t-t'}}. \tag{31}$$

Since $\arctan\sqrt{a/b} = \arcsin\sqrt{a/(a+b)}$, we get the answer

$$A(t,t') = \frac{2}{\pi}\arcsin\sqrt{\frac{2t'}{t+t'}}. \tag{32}$$

One should note that the continuum limit is valid only when $t \gg t'$ and the above expression is asymptotically true.

When $t \gg t'$, $A(t,t') \simeq \sqrt{8/\pi}(t'/t)^{1/2}$, hence $\lambda = 1$ ($\arcsin x \sim x$).

Solution 28. Droplet dynamics in the Allen-Cahn equation

Since the Laplacian in d dimensions using polar coordinate is

$$\nabla^2\phi(r) = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}\phi(r) = \frac{\partial^2\phi(r)}{\partial r^2} + \frac{d-1}{r}\frac{\partial\phi(r)}{\partial r}, \tag{33}$$

the Allen-Cahn equation becomes

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial r^2} + \frac{d-1}{r}\frac{\partial\phi}{\partial r} - V'(\phi). \tag{34}$$

Putting the ansatz into the above equation gives

$$0 = \Phi'' + \left(\frac{d-1}{r} + \frac{dR}{dt}\right)\Phi' - V'(\Phi). \tag{35}$$

Integrating the above equation after multiplying Φ' gives

$$\begin{aligned} 0 &= \frac{1}{2}\left(\Phi'(\infty)^2 - \Phi'(-\infty)^2\right) - V(\phi_0) + V(-\phi_0) + \int\left(\frac{d-1}{r} + \frac{dR}{dt}\right)(\Phi')^2 dx \\ &\simeq \text{const}\left(\frac{d-1}{R} + \frac{dR}{dt}\right), \end{aligned} \tag{36}$$

where we use that $\Phi'(x)$ is almost a delta function and $V(\phi_0) = V(-\phi_0)$. Hence the droplet radius satisfies the equation

$$\frac{dR}{dt} = -\frac{d-1}{R}. \tag{37}$$

Solution 29. Zipf's law for random texts The frequency of a given word with size ℓ is

$$x = q_s^2 q^\ell \to \ell = \frac{\ln x - 2\ln q_s}{\ln q}, \tag{38}$$

and there are m^ℓ such words. So the number of words $N(x)$ with frequency x for random texts is

$$N(x) = m^\ell = \exp\left(\frac{\ln m}{\ln q}\ln\left(\frac{x}{q_s^2}\right)\right) = \left(\frac{x}{q_s^2}\right)^{\ln m/\ln q} \sim x^{-\alpha}, \tag{39}$$

with $\alpha = -\ln m/\ln q$. When m is very large and $q_s \ll 1$ (recall that $q = (1-q_s)/m$),

$$\alpha = \frac{\ln m}{\ln m - \ln(1-q_s)} \approx 1 - \frac{q_s}{\ln m} \approx 1. \tag{40}$$