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WS 2007/2008

# Solutions in Advanced Statistical Physics

**Solution 7: Equal probability of microstates and Bernoulli measure**

$$
\begin{aligned}
\langle \eta_{i_1} \ldots \eta_{i_k} (1-\eta_{j_1}) \ldots (1-\eta_{j_\ell}) \rangle &= \frac{\binom{L-k-\ell}{N-k}}{\binom{L}{N}} \\
&= \frac{N(N-1)\cdots(N-k+1)(L-N)(L-N-1)\cdots(L-N-\ell+1)}{L(L-1)\cdots(L-k-\ell+1)} \\
&\approx \left(\frac{N}{L}\right)^k \left(1-\frac{N}{L}\right)^\ell \left[1 - \frac{k(k-1)}{2N} - \frac{\ell(\ell-1)}{2(L-N)} + \frac{(k+\ell)(k+\ell-1)}{2L}\right] \\
&= \rho^k (1-\rho)^\ell \left[1 + \frac{1}{L}\left(k\ell + \frac{1-\rho}{2\rho} k(k-1) - \frac{\rho}{2(1-\rho)}\ell(\ell-1)\right)\right] \to \rho^k(1-\rho)^\ell,
\end{aligned}
$$

where all indices are different and $k, \ell \ll N$. The stationary particle current is

$$J = p\langle \eta_i (1-\eta_{i+1})\rangle - q\langle (1-\eta_i)\eta_{i+1}\rangle = (p-q)\langle \eta_i(1-\eta_{i+1})\rangle,$$

because the order of indices does not matter. Hence the current is ($k = \ell = 1$)

$$J = (p-q)\rho(1-\rho)\left(1-\frac{1}{L}\right)^{-1} \approx (p-q)\rho(1-\rho) + \frac{(p-q)\rho(1-\rho)}{L}.$$

**Solution 8: Single file diffusion**

From the diffusion equation, we get

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} \to \frac{\partial \hat\phi}{\partial t} = -Dk^2 \hat\phi(t) \to \hat\phi(k, t-t') = e^{-Dk^2(t-t')}\hat\phi(k,t').$$

Hence density-density correlation function is

$$\langle \hat\phi(k,t)\hat\phi(k',t')\rangle = \left(\Theta(t-t')e^{-Dk^2(t-t')} + \Theta(t'-t)e^{-Dk'^2(t'-t)}\right)\langle \hat\phi(k,0)\hat\phi(k',0)\rangle,$$

where $\Theta(0) = \frac{1}{2}$ is assumed and the stationarity condition, that is, $\langle \hat\phi(k,t)\hat\phi(k',t)\rangle = \langle \hat\phi(k,0)\hat\phi(k',0)\rangle$ for arbitrary $t$, has been used. The equal time correlation function can be calculated as

$$
\begin{aligned}
\langle \hat\phi(k,0)\hat\phi(k',0)\rangle &= \int dx \int dy e^{ikx+ik'y}\langle \phi(x,0)\phi(y,0)\rangle \\
&= \int dx \int dy e^{ikx+ik'y} \bar\rho(1-\bar\rho)\delta(x-y) = L\bar\rho(1-\bar\rho)\delta_{-k,k'},
\end{aligned}
$$

which yields

$$\langle\hat{\phi}(k,t)\hat{\phi}(k',t')\rangle = L\bar{\rho}(1-\bar{\rho})e^{-Dk^2|t-t'|}\delta_{-k,k'} \equiv S(k,t-t')\delta_{-k,k'}. \tag{1}$$

Note that we are working with *discrete* Fourier components, i.e. the $k$ take discrete values compatible with a finite system of size $L$ with periodic boundary conditions.

- Method I.

  Since

  $$\langle X_0(t)^2\rangle = \int_0^t ds \int_0^t dw \langle j(0,s)j(0,w)\rangle/\bar{\rho}^2 = \int_0^t ds \int_0^t dw \sum_{k,q}\langle\hat{j}(k,s)\hat{j}(q,w)\rangle/(L^2\bar{\rho}^2),$$

  where $\hat{j}(k,t)$ is the Fourier component of $j(x,t)$, we have to find the current-current correlation function as a first step. Since $\partial_t\phi = -\partial_x j$, $\hat{j}(k,t)$ is related to $\hat{\phi}(k,t)$ such that $ik\hat{j}(k,t) = \partial_t\phi(k,t)$. Hence

  $$\begin{aligned}\langle\hat{j}(k,s)\hat{j}(q,w)\rangle &= -\frac{1}{k^2}\frac{\partial^2}{\partial s\partial w}\langle\hat{\phi}(k,s)\hat{\phi}(q,w)\rangle = \frac{L\bar{\rho}(1-\bar{\rho})}{k^2}\delta_{k,-q}\frac{\partial^2}{\partial s^2}e^{-Dk^2|s-w|}\\ &= DL\bar{\rho}(1-\bar{\rho})\left(2\delta(s-w) - Dk^2e^{-Dk^2|s-w|}\right)\delta_{k,-q},\end{aligned}$$

  where $\frac{d}{dx}|x| = 2\Theta(x)-1$, $(2\Theta(x)-1)^2 = 1$ (except $x=1$), $\partial_w g(|s-w|) = -\partial_s g(|s-w|)$, and $\frac{d}{d\tau}\Theta(\tau) = \delta(\tau)$ have been used. Hence

  $$\begin{aligned}\langle X_0(t)^2\rangle &= \frac{D(1-\bar{\rho})}{L\bar{\rho}}\sum_k \int ds \int dw \left(2\delta(s-w) - Dk^2e^{-Dk^2|s-w|}\right)\\ &= \frac{2D(1-\bar{\rho})}{L\bar{\rho}}\sum_k\int_0^t ds e^{-Dk^2 s} = \frac{2(1-\bar{\rho})}{\bar{\rho}}\frac{1}{2\pi}\int dk\frac{1-e^{-Dk^2t}}{k^2} = 2\frac{(1-\bar{\rho})}{\bar{\rho}}\sqrt{\frac{Dt}{\pi}}\end{aligned}$$

  where $\int ds\, dw\, g(|s-w|) = 2\int_0^t ds\int_0^s dw g(s-w)$ and $\frac{1}{L}\sum_k = \frac{1}{2\pi}\int dk$ were employed.

- Method II.

  Let $N(x,t) = \int_0^t dsj(x,s)$. Note that this differs somewhat from the (incorrect) definition (6) in the problem set. Then $\langle X_0(t)^2\rangle = \langle(N(0,t))^2\rangle/\bar{\rho}^2$ and

  $$\frac{\partial N}{\partial x} = \int_0^t ds\frac{\partial j(x,s)}{\partial x} = -\int_0^t ds\frac{\partial\phi(x,s)}{\partial s} = \phi(x,0) - \phi(x,t),$$

  where the continuity equation has been used. The Fourier components of $N$ can be written as $ik\hat{N}(k,t) = \hat{\phi}(k,t) - \hat{\phi}(k,0)$. So,

  $$\begin{aligned}\langle X_0(t)^2\rangle &= \frac{1}{L^2}\sum_{k,q}\langle\hat{N}(k,t)\hat{N}(q,t)\rangle\\ &= -\frac{1}{L^2}\sum_{k,q}\frac{1}{kq}\langle(\hat{\phi}(k,t)-\hat{\phi}(k,0))(\hat{\phi}(q,t)-\hat{\phi}(q,0))\rangle\\ &= \frac{2\bar{\rho}(1-\bar{\rho})}{2\pi}\int dk\frac{1-e^{-Dk^2t}}{k^2} = 2\frac{(1-\bar{\rho})}{\bar{\rho}}\sqrt{\frac{Dt}{\pi}},\end{aligned}$$

where we used Eq. (1) and the stationarity condition.

As $\bar{\rho} \to 0$, the variance diverges but this has to be expected because in this limit the tracer particle does not feel the other particles and the variance should be $\sim t$ which is much larger than $\sqrt{t}$.

**Solution 9: The zero range process**

a.) The detailed balance condition means

$$\mathrm{P}^*[n_1, \ldots, n_l, n_{l+1}, \ldots, n_L]\gamma(n_l) = \mathrm{P}^*[n_1, \ldots, n_l - 1, n_{l+1} + 1, \ldots, n_L]\gamma(n_{l+1} + 1), \tag{2}$$

which, combined with the product measure property, yields

$$\gamma(n_l)f(n_l)f(n_{l+1}) = \gamma(n_{l+1}+1)f(n_l-1)f(n_{l+1}+1) \to \gamma(n_l)\frac{f(n_l)}{f(n_l-1)} = \gamma(n_{l+1}+1)\frac{f(n_{l+1}+1)}{f(n_{l+1})}.$$

Since the term on the left (right) hand side is a function of only $n_l$ $(n_{l+1})$, the above term should be a constant, say $\alpha$. Hence

$$f(n) = f(0)\alpha^n \prod_{k=1}^{n} \gamma(k)^{-1} \tag{3}$$

which corrects Eq.(7) of the problem set. The free parameters $\alpha$ and $f(0)$ are fixed by the normalization and the mean particle number per site [provided the corresponding series converge, see part c.)].

b.) The master equation reads

$$\frac{\partial}{\partial t}\mathrm{P}[\{n\};t] =$$

$$\sum_{k=1}^{L} \left(p\gamma(n_{k-1}+1)\mathrm{P}_{k-1,k}[\{n\};t] + (1-p)\gamma(n_{k+1}+1)\mathrm{P}_{k+1,k}[\{n\};t] - \gamma(n_k)\mathrm{P}[\{n\};t]\right), \tag{4}$$

where $\{n\} \equiv n_1, \ldots, n_L$ and

$$\mathrm{P}_{i,j}[\{n\};t] \equiv \mathrm{P}[\{m\};t]|_{m_i=n_i+1, m_j=n_j-1, m_k=n_k (k\neq i,j)}.$$

We have to verify that upon inserting the product measure solution of part a.) the right hand side of (4) vanishes. The product measure property implies that

$$\mathrm{P}^*_{i,j}[\{n\}] = \mathrm{P}^*[\{n\}]\frac{f(n_i+1)f(n_j-1)}{f(n_i)f(n_j)}.$$

and hence, using the results of part a.),

$$\gamma(n_{k-1}+1)\mathrm{P}^*_{k-1,k}[\{n\}] = \alpha\frac{f(n_k-1)}{f(n_k)}\mathrm{P}^*[\{n\}] = \gamma(n_k)\mathrm{P}^*[\{n\}]$$

and similarly $\gamma(n_{k+1}+1)\mathrm{P}^*_{k+1,k}[\{n\}] = \gamma(n_k)\mathrm{P}^*[\{n\}]$. Thus the product measure is the stationary solution for any $p$.

c.) The convergence test says that if

$$\frac{f(n)}{f(n+1)} = \frac{\alpha}{\gamma(n)}$$

becomes larger (smaller) than 1 with increasing $n$, the series $\sum_n f(n)$ converges (diverges). If the ratio converges to a finite number, then by adjusting the parameter $\alpha$ we can always make sure that $\frac{f(n)}{f(n+1)} \to 1$ without loss of generality. In this case, the Gauss test will determine the convergence of the series. Let us assume that

$$\frac{f(n)}{f(n+1)} = 1 + \frac{b}{n} + \frac{B(n)}{n^2}$$

where $b$ is a constant and $B(n)$ is bounded for large $n$. If $b > 1$ ($b \leq 1$) then the series converges (diverges).