Notions of probability:

- objective → frequency
- subjective → likelihood, measure of belief (Bayes)

1° Basic concepts

a) Random variables

Mathematically, a probability space is a triple $\{\Omega, \mathcal{F}, P\}$ with

- Ω : set of all possible outcomes
- $\mathcal{F}$: set of events, i.e. subsets of Ω (a σ-algebra)
- P : probability measure $P : \mathcal{F} \to [0,1]$ such that
  - $P(A \cup B) = P(A) + P(B)$ for disjoint events A, B, and
  - $P(\Omega) = 1$

Two standard settings are of interest here:

(i) Ω countable, e.g. $\Omega = \mathbb{N}, \mathbb{Z}$

$\Rightarrow X \in \Omega$ is a discrete random variable

(ii) $\Omega = \mathbb{R}$, $\mathcal{F}$ set of intervals of $\mathbb{R}$,

$$P([a,b]) = \int_a^b dx\, f(x)$$

$f(x)$: probability density function (pdf)

$$F(x) = P((-\infty, a]) = \text{Prob}[X \leq a] =$$
$$= \int_{-\infty}^{a} dx\, f(x)$$

is the probability distribution function or cumulative distribution.

In the following we focus a setting (ii) and its higher-dimensional generalization.

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b) Expectation values

Expectation value of a function $\varphi(X)$ of a RV X is

$$\underset{\text{math.}}{\mathbb{E}(\varphi(X))} = \underset{\text{phys.}}{\langle \varphi(X) \rangle} = \int dx\, \varphi(x)\, f(x)$$

$$\left.\begin{array}{ll} \underline{\text{Moments:}} & \mu_n = \langle X^n \rangle \\ \underline{\text{Mean:}} & \mu_1 \\ \underline{\text{Variance:}} & \sigma^2 = \mu_2 - \mu_1^2 \end{array}\right\}$$

Characteristic fct. / moment generating fct.:

$$G_X(k) = \langle e^{ikX} \rangle = \int dx \, e^{ikx} f(x)$$

$$G_X(0) = 1 \, , \quad |G_X(k)| \le 1$$

$$\Rightarrow \quad G_X(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mu_n \, ,$$

$$\mu_n = \left(\frac{1}{i}\right)^n \frac{d^n}{dk^n} G_X(k) \Big|_{k=0}$$

Cumulants: (motivation later)

$$\ln G_X(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \underline{\kappa_n}$$

$$\frac{d}{dk} \ln G_X(k) \Big|_{k=0} = \frac{G_X'(0)}{G_X(0)} = \mu_1$$

$$\frac{d^2}{dk^2} \ln G_X(k) \Big|_{k=0} = \frac{G_X''(0)}{G_X(0)} - \frac{\left(G_X'(0)\right)^2}{G_X(0)^2} =$$

$$= \mu_2 - \mu_1^2 = \sigma^2$$

Note:
- Moments do not always uniquely determine the distribution.
- Not all moments necessarily exist

c) Dependence and correlation

Consider two events A, B. Then the conditional probability of observing B given A is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A \vee B)}{P(A)}$$

B is independent of A if

$$P(B|A) = P(B) \Rightarrow$$

$$\Rightarrow \quad P(A \vee B) = P(A)\,P(B)$$

Bayes theorem:

$$P(A|B) = \frac{P(A \vee B)}{P(B)} = \frac{P(B|A)}{P(B)} \cdot P(A)$$

shows how a priori-probability P(A) is modified by the observation of B, if P(B|A) and P(B) are known.

Application: Inference of P(A|B) from P(B|A).

Similarly for independent RV's X_1, X_2 we have

$$\left.\begin{aligned} f_{12}(x_1, x_2) &= f_1(x_1)\, f_2(x_2) \\ f_{1|2}(x_1 | x_2) &:= \frac{f_{12}(x_1, x_2)}{f_2(x_2)} = f_1(x_1) \end{aligned}\right\}$$

and for the corresponding generating fcts

$$G_{12}(k_1, k_2) := \langle e^{i(k_1 X_1 + k_2 X_2)} \rangle = G_1(k_1)\, G_2(k_2)$$

The covariance of X_1, X_2 is defined by

$$\left.\begin{aligned} \mathrm{Cov}(X_1, X_2) &= \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle \\ &= \langle X_1 X_2 \rangle_c \quad \text{in physics notation} \end{aligned}\right\}$$

Independence implies $\mathrm{Cov}(X_1, X_2) = 0$, but the converse is not true:

(i) $X_2 = X_1^2$, $f_1(x_1) = f_1(-x_1)$ symmetric

$$\Rightarrow \langle X_1 X_2 \rangle = \langle X_1^3 \rangle = 0 = \langle X_1 \rangle \langle X_2 \rangle$$

(ii) Increments of stock prices are (to good approximation) uncorrelated, but volatilities are not.

Figures

d) Sums of random variables

X_1, X_2 RV's with pdf $f_{12}(x_1,x_2)$, $Y = X_1 + X_2$

$$\Rightarrow f_Y(y) = \int dx_1 \int dx_2\, f_{12}(x_1,x_2)\, \delta(x_1+x_2-y) =$$
$$= \int dx_1\, f_{12}(x_1, y-x_1) \underset{\text{independence}}{=} \int dx_1\, f_1(x_1) f_2(y-x_1)$$

and $G_Y(k) = \langle e^{ik(X_1+X_2)} \rangle \overset{\text{independence}}{=} G_1(k)\, G_2(k)$

$$\Rightarrow \ln G_Y(k) = \ln G_1(k) + \ln G_2(k)$$

As a consequence, all cumulants of X_1 and X_2 add up to cumulants of Y:

$$\left.\begin{aligned} \langle Y \rangle &= \langle X_1 \rangle + \langle X_2 \rangle \\ \sigma_Y^2 &= \sigma_1^2 + \sigma_2^2 \qquad \text{etc.} \end{aligned}\right\}$$

e) Transformation of random variables

Consider RV X, $Y = \varphi(X)$, then in general

$$\left.\begin{aligned} f_Y(y) &= \int dx\, f_X(x)\, \delta(y - \varphi(x)) \\ G_Y(k) &= \langle e^{ik\varphi(X)} \rangle_X \end{aligned}\right\}$$

In particular, if φ is monotonically increasing ($\varphi'(x) > 0$) then

$$F_Y(y) = \text{Prob}[Y \le y] = \text{Prob}[X \le x] = F_X(x) \quad \text{with } y = \varphi(x)$$

$$\Rightarrow \quad F_Y(\varphi(x)) = F_X(x)$$

$$\Rightarrow \quad f_X(x) = \frac{d}{dx} F_X(x) = \varphi'(x) \frac{d}{dy} F_Y(y) = \varphi'(x) f_Y(y)$$

$$\Rightarrow \quad f_Y(y) = \varphi'(x)^{-1} f_X(x) = \left(\frac{dy}{dx}\right)^{-1} f_X(x)$$

Similarly if $\varphi'(x) < 0$ we have

$$F_Y(\varphi(x)) = 1 - F_X(x)$$

$$\Rightarrow \quad f_Y(y) = -\varphi'(x)^{-1} f_X(x)$$

$\Rightarrow$ generally: $f_Y(y) = |\varphi'(x)|^{-1} f_X(x)$

f) Gaussian RV's

- one-dim Gaussian: $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\Rightarrow \quad G(k) = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x) = e^{ik\mu_1 - \frac{1}{2}\sigma^2 k^2}$$

$$\left.\begin{array}{l} \Rightarrow \quad \ln G(k) = ik\mu_1 - \frac{1}{2}\sigma^2 k^2 \\ \text{all higher cumulants vanish} \end{array}\right\}$$

This property is obviously preserved under addition

$$\left.\begin{array}{l} \Rightarrow \quad \text{sums of independent Gaussian RVs} \\ \text{are Gaussian} \end{array}\right\}$$

- multi-dim. Gaussian: $\vec{X} = (X_1, .., X_N)$

  then the general form of a Gaussian pdf reads

$$f(x_1, .., x_N) = \frac{1}{Z} \exp\left(-\frac{1}{2} \vec{x} \cdot \hat{A} \cdot \vec{x} - \vec{B} \cdot \vec{x} \right)$$

$$\left.\begin{array}{l} \hat{A} = \{ A_{ij} \}_{i,j=1,..,N} \quad \text{symmetric, positive def.} \\ \frac{1}{Z} = (2\pi)^{-N/2} (\det \hat{A})^{1/2} \exp\left(-\frac{1}{2} \vec{B} \cdot \hat{A}^{-1} \cdot \vec{B} \right) \end{array}\right\}$$

Mean values and covariances are then given by

$$\left.\begin{aligned} \langle X_i \rangle &= -\sum_j (\hat{A}^{-1})_{ij} B_j \\ \mathrm{Cov}(X_i, X_j) &= (\hat{A}^{-1})_{ij} \end{aligned}\right\}$$

Corollary: Uncorrelated Gaussian RV's are also independent.

Proof: $X_1, \ldots, X_N$ uncorrelated $\Rightarrow$ $\hat{A}^{-1}$ diagonal $\Rightarrow$ $\hat{A}$ diagonal $\Rightarrow$ $X_1, \ldots, X_N$ independent. $\square$

2° Limit laws

Consider i.i.d. RV's $X_1, \ldots, X_N$, pdf $f(x)$.
What can we say about the sum

$$S_N := \sum_{i=1}^{N} X_i \quad \text{for } N \to \infty \, ?$$

The law of large numbers concerns the behavior of S_N itself: