Summary of stochastic processes encountered so far

Process	stationary	continuous	Markovian	Gaussian
Poisson	no	no	yes	no
telegraph	yes	no	yes	no
Wiener	no	yes	yes	yes
Ornstein-Uhlenbeck	yes	yes	yes	yes
Lévy	no	no	yes	no
FBM	no	yes	no	yes

IV. Stochastic differential equations

1° Langevin equations

a) White noise and the Wiener process

Consider a stationary Gaussian process $\xi(t)$ with $\langle \xi \rangle = 0$ and autocorrelation $K(\tau)$.
Define the stochastic process $Y(t)$ through

$$\frac{dY}{dt} = \xi(t), \qquad Y(0) = 0$$

$$\Rightarrow \quad Y(t) = \int_0^t ds\, \xi(s), \qquad \langle Y \rangle = 0 \quad \text{and}$$

$$\langle Y(t) Y(t') \rangle = \int_0^t ds \int_0^{t'} ds'\, K(s-s')$$

To be specific, take ξ to be Gaussian & Markovian such that

$$K(\tau) = \sigma^2 e^{-|\tau|/\tau_c}$$

$$\Rightarrow \langle Y(t) Y(t') \rangle = \sigma^2 \int_0^t ds \int_{-s}^{t'-s} d\tau\, e^{-|\tau|/\tau_c} =$$

$$= 2\sigma^2 \tau_c \min(t,t') + \sigma^2 \tau_c^2 \left[e^{-t'/\tau_c} + e^{-t/\tau_c} - 1 - e^{-|t-t'|/\tau_c} \right]$$

Now we take $\tau_c \to 0$, $\sigma^2 \to \infty$ such that

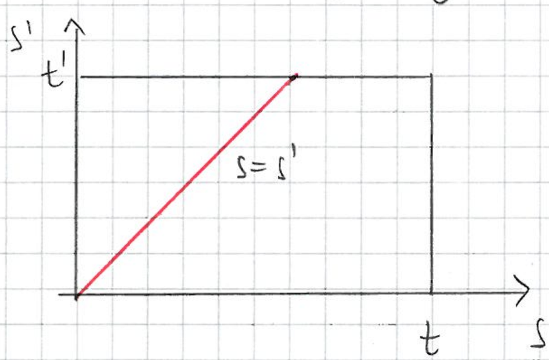
$K(\tau) \to \delta(\tau)$ and $\xi(t)$ becomes white noise. This requires

$$\int_{-\infty}^{\infty} d\tau \, K(\tau) = 2\sigma^2 \tau_c = 1$$

$$\left.\begin{array}{l} \Rightarrow \sigma^2 \tau_c^2 \to 0 \quad \text{and} \\ \langle Y(t) Y(t') \rangle \to \min(t, t') \quad \text{Wiener process} \end{array}\right\}$$

This result can also be derived directly:

$$\langle Y(t) Y(t') \rangle = \int_0^t ds \int_0^{t'} ds' \, \delta(s-s') = \min(t, t')$$

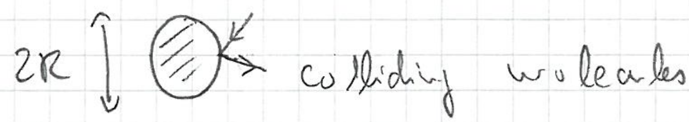


Remarks:

- White noise is the derivative of the (non-differentiable) Wiener process.
- In applications, white noise should always be treated as a limit of a stationary Gaussian Markov-process with correlation time $\tau_c \to 0$
- Ideally the limit should be performed at the end of the calculation, but this is impractical.

b) Langevin's theory of Brownian motion (1906)

Consider a particle of radius R, mass m, immersed in a fluid.



Equation of motion for one velocity component:

$$\text{(L)} \quad m\dot{v} + \gamma v = \xi(t) \leftarrow \text{stochastic Langevin force}$$

Stokes friction, $\gamma = 6\pi\eta R$

We assume that the correlation time of the molecular collisions is very short and treat $\xi(t)$ as white noise:

noise amplitude

$$\langle \xi \rangle = 0, \quad \langle \xi(t)\,\xi(t') \rangle = A\,\delta(t-t')$$

The solution of (L) with initial condition $v(0)=0$ reads

$$v(t) = \frac{1}{m}\int_0^t ds\; e^{-\gamma/m\,(t-s)}\,\xi(s)$$

$$\Rightarrow \langle v(t)\,v(t') \rangle = \frac{1}{m^2}\int_0^t ds \int_0^{t'} ds'\; e^{-\frac{\gamma}{m}(t-s)}\, e^{-\frac{\gamma}{m}(t'-s')}\, A\,\delta(s-s')$$

$$= \frac{A}{2\gamma m}\left(e^{-\frac{\gamma}{m}|t-t'|} - e^{-\frac{\gamma}{m}(t+t')} \right)$$

$$\xrightarrow[t,t' \to \infty]{} \underline{\frac{A}{2\gamma m}\, e^{-\frac{\gamma}{m}|t-t'|}}$$

Ornstein-Uhlenbeck process

Now the noise amplitude can be fixed by thermodynamics:

$$\text{kinetic energy} = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k_B T$$

equipartition thm

$$\Rightarrow \quad \frac{A}{2\gamma m} = \frac{k_B T}{m} \quad \Rightarrow \quad \underline{A = 2\gamma k_B T}$$

The velocity correlation time

$$\tau_c = \frac{m}{\gamma} \sim 10^{-8}\ \mathrm{s}$$

$\Rightarrow$ replace v by white noise with

$$\sigma^2 \tau_c = \frac{A}{2\gamma m} \cdot \frac{m}{\gamma} = \frac{k_B T}{\gamma}$$

$\Rightarrow$ particle position is a Wiener process with

$$\langle x(t)\, x(t') \rangle = \frac{2 k_B T}{\gamma} \min(t, t')$$

convention

$$\langle (x(t) - x(t'))^2 \rangle = \frac{2 k_B T}{\gamma} |t - t'| = 2D |t - t'|$$

with the diffusion coefficient

$$\underline{D = \frac{k_B T}{\gamma} = \frac{k_B T}{6\pi \eta R}} \qquad \text{Einstein relation}$$

c) The Langevin approach

To summarize, we have shown that the Wiener process is the solution of

$$\frac{dY}{dt} = \xi(t), \quad \langle \xi(t)\,\xi(t')\rangle = \delta(t-t')$$

Similarly the Ornstein-Uhlenbeck process is the solution of

$$\frac{dY}{dt} = -Y + \xi(t), \quad \langle \xi(t)\,\xi(t')\rangle = 2\,\delta(t-t')$$

In the following we consider general (one-dimensional) Langevin equations of the form

$$\frac{dY}{dt} = f(Y) + g(Y)\,\xi(t) \qquad (*)$$

where $\xi(t)$ is white noise.

- The noise is called additive if $g = \text{const.}$, and multiplicative otherwise.
- The process defined by $(*)$ is always Markovian but generally non-Gaussian.
- To characterize the process defined (?) by $(*)$ we need to derive an equation for the transition probability T_τ of the process.

2° The Fokker-Planck equation

a) Differential Chapman-Kolmogorov-equation

Consider a homogeneous Markov process with transition probability $T_\tau(y_2|y_1)$.

How does T_τ behave for $\tau \to 0$?

Two cases:

(i) Jump processes: For $\tau \to 0$ there is at most one jump in the time interval $[0, \tau]$

$$\Rightarrow \quad T_\tau(y_2|y_1) = \underbrace{(1 - a_0 \tau)\,\delta(y_2 - y_1)}_{\text{prob. of no jump}} + \tau\, W(y_2|y_1) + O(\tau^2)$$

with the jump rate

$$W(y_2|y_1) = \lim_{\tau \to 0} \frac{1}{\tau} T_\tau(y_2|y_1) \quad , \quad y_2 \neq y_1$$

Example: Poisson process

$$T_\tau(N_2|N_1) = e^{-\rho\tau} \frac{(\rho\tau)^{N_2 - N_1}}{(N_2 - N_1)!} \quad , \quad N_2 \geq N_1$$

$$\simeq (1 - \rho\tau)\,\delta_{N_1, N_2} + \rho\tau\,\delta_{N_1, N_2 - 1} + O(\tau^2)$$

Inserting into the CK-equation yields

$$T_{\tau + \Delta\tau}(y_3|y_1) = \int dy_2\, T_{\Delta\tau}(y_3|y_2)\, T_\tau(y_2|y_1) =$$

$$= \int dy_2 \left[(1 - a_0(y_2)\Delta\tau)\, \delta(y_3 - y_2) + \Delta\tau\, W(y_3|y_2) \right] \times$$
$$\times\, T_\tau(y_2|y_1) =$$

$$= (1 - a_0(y_3)\Delta\tau)\, T_\tau(y_3|y_1) +$$
$$+ \Delta\tau \int dy_2\, W(y_3|y_2)\, T_\tau(y_2|y_1)$$

and $a_0(y) = \int dy'\, W(y'|y)$ by normalization

$$\Rightarrow \frac{\partial}{\partial\tau} T_\tau(y_3|y_1) = \int dy_2 \left[W(y_3|y_2)\, T_\tau(y_2|y_1) - W(y_2|y_3)\, T_\tau(y_3|y_1) \right] \quad \text{(M)}$$

Master equation

(ii) Continuous processes: In this case we know that

$$\lim_{\tau \to 0} \frac{1}{\tau} T_\tau(y_2|y_1) = 0 \quad \text{whenever } |y_2 - y_1| > \varepsilon$$

which implies that the displacement within a time interval of length τ vanishes with $\tau \to 0$.
In this case the short-time behavior of T_τ is characterized through the moments

$$\left.\begin{aligned} a_1(y) &:= \lim_{\tau \to 0} \frac{1}{\tau} \int dx\, (x - y)\, T_\tau(x|y) \\ a_2(y) &= \lim_{\tau \to 0} \frac{1}{\tau} \int dx\, (x-y)^2\, T_\tau(x|y) \end{aligned}\right\}$$

Example: Wiener process

$$T_\tau(y_2|y_1) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y_2-y_1)^2}{2\tau}}$$

$\Rightarrow$ $a_1(y) = 0, \quad a_2(y) = 1$

and higher moments vanish.

To derive a differential equation for T_τ in this case we consider the time evolution of the expectation value of a test function ϕ:

$$\frac{d}{d\tau} \int dx\, \phi(x)\, T_\tau(x|y) =$$

$$= \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \int dx\, \phi(x) \left[T_{\tau+\Delta\tau}(x|y) - T_\tau(x|y) \right] =$$

$$= \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \left[\int dx \int dz\, \phi(x)\, T_{\Delta\tau}(x|z)\, T_\tau(z|y) - \right.$$

$$\left. - \int dx\, \phi(x)\, T_\tau(x|y) \right] = \quad \text{(relabel)}$$

$$= \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \int dx \int dz\, \underbrace{\left(\phi(x) - \phi(z)\right)}_{\simeq\, \phi'(z)(x-z) + \frac{1}{2}\phi''(z)(x-z)^2} T_{\Delta\tau}(x|z)\, T_\tau(z|y)$$

$$= \int dz \left(\phi'(z)\, a_1(z) + \frac{1}{2} \phi''(z)\, a_2(z) \right) T_\tau(z|y) =$$

[integrate by parts]

$$= \int dz\, \phi(z) \left(- \frac{\partial}{\partial z} a_1(z) + \frac{1}{2} \frac{\partial^2}{\partial z^2} a_2(z) \right) T_\tau(z|y)$$

$$\Rightarrow \frac{\partial}{\partial \tau} T_\tau(x|y) = \left(- \frac{\partial}{\partial x} a_1 + \frac{1}{2} \frac{\partial^2}{\partial x^2} a_2 \right) T_\tau(x|y)$$

Fokker-Planck equation (FP)

A general Markov process may have both continuous and discontinuous contributions (e.g. the Cauchy process). In this case the differential Chapman-Kolmogorov equation is a combination of (M) and (FP).

Remark: Since $P_1(y,t) = \int dy'\, T_t(y|y')\, P_1(y',0)$, (M) and (FP) can also be viewed as evolution equations for P_1.

Examples:

- Wiener process $T_\tau(y|y') = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(y-y')^2}{2\tau}}$

$$\Rightarrow \frac{\partial}{\partial \tau} T_\tau = \left[- \frac{1}{2\tau} + \frac{(y-y')^2}{2\tau^2} \right] T_\tau$$

$$\frac{\partial}{\partial y} T_\tau = - \frac{(y-y')}{\tau} T_\tau$$

$$\frac{\partial^2}{\partial y^2} T_\tau = \left[- \frac{1}{\tau} T_\tau + \frac{(y-y')^2}{\tau^2} \right] T_\tau = 2 \frac{\partial T_\tau}{\partial \tau}$$

$\square$

- Ornstein-Uhlenbeck process:

$$T_\tau = \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}} \exp\left[-\frac{(y-y'e^{-\tau})^2}{2(1-e^{-2\tau})}\right]$$

satisfies FP with $a_1 = -y, \quad a_2 = 2$ Problem

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b) Kramers-Moyal expansion

Apart from its fundamental role for the description of continuous processes, the FP is often used as an approximation of the master equation M. This approximation can be justified systematically if there is a scaling parameter in the problem under which the jumps become "small".

Here we simply assume that the jump rates

$$W(y|y') = w(y', \underbrace{y-y'}_{r})$$

are strongly localized in r but slowly varying in y'. Then we can expand the RHS of M as follows:

$$\frac{\partial}{\partial t} P_1(y,t) = \int dr \; \underbrace{w(y-r,r)\, P_1(y-r,t)}_{} -$$

$$= w(y,r)\, P_1(y,t) - r\frac{\partial}{\partial y} w(y,r) P_1 + \frac{1}{2} r^2 \frac{\partial^2}{\partial y^2} w\, P_1 + \ldots$$

$$- \int dr\, w(y,r)\, P_1(y,t) =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial y^n} \underbrace{\left(\int dr\, r^n\, w(y,r) \right)}_{a_n(y) \text{ jump moment}} P_1(y,t)$$

Kramers - Moyal expansion

Truncation to second order yields FP

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c) Stationary distribution

If the process of interest is stationary, then

$$\frac{\partial P_1}{\partial t} = \underbrace{\left(- \frac{\partial}{\partial y} a_1(y) + \frac{1}{2} \frac{\partial^2}{\partial y^2} a_2(y) \right)}_{\text{FP - Operator } \mathcal{L}} P_1 = 0$$

if P_1 is the stationary distribution $P_s(y)$.

Thus P_s is a right eigenfunction of $\mathcal{L}$ with eigenvalue 0.

Noting that FP can be written as a continuity equation

$$\frac{\partial P_1}{\partial t} + \frac{\partial J}{\partial y} = 0.$$

with the probability current

$$J = a_1(y) P_1 - \frac{1}{2} \frac{\partial}{\partial y} a_2(y) P_1$$

we see that $\mathcal{L} P_s = 0$ amounts to

$$J_s = a_1 P_s - \frac{1}{2} \frac{\partial}{\partial y} a_2(y) P_s = \text{const.} = 0$$

because the probability current has to vanish at the boundaries of the system (or at infinity). Thus P_s is the solution of

$$\frac{1}{2} \frac{d}{dy} \underbrace{a_2(y) P_s(y)}_{= Q(y)} = a_1(y) P_s(y)$$

$$\Rightarrow \frac{dQ}{dy} = 2 \frac{a_1}{a_2} Q(y) \Rightarrow Q = C \exp\left[2 \int_0^y dy' \frac{a_1}{a_2} \right]$$

$$\Rightarrow P_s(y) = \frac{C}{a_2(y)} \exp\left[2 \int_0^y dy' \frac{a_1(y')}{a_2(y')} \right]$$

provided this exists and can be normalized.

<u>Example:</u> Ornstein-Uhlenbeck process

$a_1 = -y$, $a_2 = 2 \Rightarrow P_s(y) \sim \exp\left(-\frac{y^2}{2}\right)$