3° Equivalence of Langevin- and Fokker-Planck equations

We have found two equivalent, differential representations of the Ornstein-Uhlenbeck process:

(i) Langevin: $\dot{Y} = -Y + \xi(t)$, $\langle \xi(t)\xi(t')\rangle = 2\delta(t-t')$

(ii) Fokker-Planck:

$$\frac{\partial}{\partial t} P_1(y,t) = \frac{\partial}{\partial y}(y P_1) + \frac{\partial^2}{\partial y^2} P_1$$

Which FP-equation corresponds to the general Langevin-equation

$$\dot{Y} = f(Y) + g(Y)\,\xi(t) \quad ?$$

a) Additive noise:
$$\left.\begin{aligned} \dot{Y} &= f(Y) + \xi(t) \\ \langle \xi(t)\xi(t')\rangle &= A\,\delta(t-t') \end{aligned}\right\}$$

We need to compute the moments

$$\left\{\begin{aligned} a_k(y) &= \lim_{\tau\to 0} \frac{1}{\tau} \langle (\underbrace{Y(t+\tau) - Y(t)}_{\Delta Y})^k \rangle, \quad k = 1,2 \\ \Delta Y &= Y(t+\tau) - Y(t) = \int_t^{t+\tau} ds\, f(Y(s)) + \int_t^{t+\tau} ds\, \xi(s) \end{aligned}\right.$$

$$\Rightarrow \langle \Delta Y \rangle = \int_t^{t+\tau} ds\, f(Y(s)) \simeq \tau \cdot f(Y(t)) + \mathcal{O}(\tau^2)$$

$$\Rightarrow \underline{a_1(y) = f(y)}$$

$$\langle (\Delta Y)^2 \rangle = \underbrace{\int_t^{t+\tau} ds \int_t^{t+\tau} ds' \langle f(Y(s)) f(Y(s')) \rangle}_{\mathcal{O}(\tau^2)} +$$

$$+ 2\int_t^{t+\tau} ds \int_t^{t+\tau} ds' \langle f(Y(s)) \xi(s') \rangle +$$

$$+ \underbrace{\int_t^{t+\tau} ds \int_t^{t+\tau} ds' \underbrace{\langle \xi(s) \xi(s') \rangle}_{A\,\delta(s-s')}}$$

$$= \underline{A \cdot \tau}$$

<u>mixed term:</u> $\quad f(Y(s)) \simeq f(Y(t)) + f'(Y(t))[Y(s) - Y(t)]$

Moreover, at short times the process behaves as Brownian motion:

$$Y(s) - Y(t) \approx \int_t^{s} ds' \, \xi(s')$$

$$\Rightarrow \int_t^{t+\tau} ds \int_t^{t+\tau} ds' \langle f(Y(s)) \xi(s') \rangle \simeq f'(Y(t)) \times$$

$$\times \int_t^{t+\tau} ds \int_t^{t+\tau} ds' \langle (Y(s) - Y(t)) \, \xi(s') \rangle$$

$$\langle (Y(s) - Y(t))\, \xi(s') \rangle = \int_t^s ds'' \langle \xi(s'')\, \xi(s') \rangle =$$

$$= \begin{cases} A & t < s' < s \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \int_t^{t+\tau} ds \int_t^{t+\tau} ds' \langle f(Y(s))\, \xi(s') \rangle = f'(Y(t)) \cdot A \underbrace{\int_t^{t+\tau} ds\,(s-t)}_{= \frac{1}{2}\tau^2}$$

$\Rightarrow$ mixed term is $O(\tau^2)$ and therefore

$$a_2 = \lim_{\tau \to 0} \frac{1}{\tau} \langle (\Delta Y)^2 \rangle = A$$

The Fokker-Planck equation in the additive case is simply

$$\frac{\partial}{\partial t} P_1(y,t) = -\frac{\partial}{\partial y} f(y)\, P_1 + \frac{1}{2} A \frac{\partial^2}{\partial y^2} P_1$$

Stationary distribution: $\displaystyle \int_0^y dy' \frac{a_1(y')}{a_2(y')} = \frac{1}{A} \underbrace{\int_0^y dy' f(y')}_{-V(y)}$

$$\Rightarrow P_s(y) \sim \exp\left[-\frac{2}{A} V(y)\right]$$

"potential" with $f(y) = -V'(y)$

Example: For the velocity of a Brownian particle we have

$$f(v) = -\frac{\gamma}{m} v, \quad A = \frac{2\gamma k_B T}{m^2}$$

$$\Rightarrow \quad V(y) = \frac{1}{2}\frac{\gamma}{m} v^2$$

$$\Rightarrow \quad P_s(y) \sim \exp\Big[\underbrace{-\frac{m^2}{\gamma k_B T}\cdot\frac{1}{2}\frac{\gamma}{m}v^2}_{-\frac{1}{k_B T}\frac{m}{2}v^2}\Big]$$

Boltzmann distribution

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b) Multiplicative noise:

$$\dot{Y} = f(Y) + g(Y)\cdot\xi(t), \quad \langle \xi(t)\xi(t')\rangle = \delta(t-t')$$

We assume $g(Y) \neq 0$, e.g. $g > 0$ everywhere.

Rather than computing a_1, a_2 directly, we map the multiplicative case to the additive case.

Variable transformation: $\tilde{Y}(t) := \int_0^{Y(t)} dy\, g^{-1}(y)$

$$\Rightarrow \quad \frac{d\tilde{Y}}{dt} = g^{-1}(Y)\frac{dY}{dt} = \underbrace{\frac{f(Y)}{g(Y)}}_{\tilde{f}(y)} + \xi(t)$$

requires differentiability of the process ?

The distribution of the transformed variable is related to the original distribution through

$$\tilde{P}_1(\tilde{y},t) = \left|\frac{dy}{d\tilde{y}}\right| P_1(y,t) = g(y)\, P_1(y,t)$$

We know that $\tilde{P}_1$ satisfies the additive FP-eq.

$$\left.\begin{aligned} &\frac{\partial \tilde{P}_1}{\partial t} = -\frac{\partial}{\partial \tilde{y}} \tilde{f}(\tilde{y})\, \tilde{P}_1 + \frac{1}{2}\frac{\partial^2}{\partial \tilde{y}^2}\tilde{P}_1 \\ &\frac{\partial}{\partial \tilde{y}} = g\frac{\partial}{\partial y}, \quad \tilde{f}\tilde{P}_1 = f P_1 \end{aligned}\right\}$$

$$\Rightarrow \cancel{g}\frac{\partial P_1}{\partial t} = -\cancel{g}\frac{\partial}{\partial y} f P_1 + \frac{1}{2}\cancel{g}\frac{\partial}{\partial y} g \frac{\partial}{\partial y} g P_1$$

$$\Rightarrow \frac{\partial P_1}{\partial t} = -\frac{\partial}{\partial y} f(y)\, P_1 + \frac{1}{2}\frac{\partial}{\partial y} g \frac{\partial}{\partial y} g P_1$$

$$= -\frac{\partial}{\partial y} a_1(y)\, P_1 + \frac{1}{2}\frac{\partial^2}{\partial y^2} a_2(y)\, P_1$$

With $\frac{\partial}{\partial y} g^2 P_1 = g' g P_1 + g \frac{\partial}{\partial y} g P_1$

$$\Rightarrow \frac{\partial}{\partial y} g \frac{\partial}{\partial y} g P_1 = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} g^2 P_1 - g' g P_1\right]$$

we identify

$$a_1(y) = f(y) + \frac{1}{2}\, g(y)\, g'(y)$$

$$a_2(y) = g^2(y)$$

What is the origin of the "noise-induced drift" $\frac{1}{2} g g'$?

We start directly from the Langevin equation:

$$Y(t+\tau) - Y(t) = \underbrace{\int_t^{t+\tau} ds\, f(Y(s))}_{\to\ \tau\, F(Y(t))} + \underbrace{\int_t^{t+\tau} ds\, g(Y(s))\, \xi(s)}_{?}$$

Because of the strong fluctuations of white noise, the second term has no unique limit for $\tau \to 0$. We expect

$$\int_t^{t+\tau} ds\, g(Y(s))\, \xi(s) \xrightarrow[\tau \to 0]{} g(Y^*) \underbrace{\int_t^{t+\tau} ds\, \xi(s)}_{\text{well-defined as difference of Wiener process}}$$

for some Y^* between $Y(t)$ and $Y(t+\tau)$.
Unfortunately, the result depends on the choice of Y^*

$\Rightarrow$ <u>multiplicative Langevin-eqs. require a interpretation</u>

Itô interpretation (1944):

$$Y^* = 0 \Rightarrow \int_t^{t+\tau} ds\, g(Y(s))\, \xi(s) \to g(Y(t)) \int_t^{t+\tau} ds\, \xi(s)$$

Stratonovich interpretation (1966):

$$Y^* = \frac{1}{2}\left(Y(t) + Y(t+\tau)\right)$$

General: $\underline{Y^* = (1-\alpha)\, Y(t) + \alpha\, Y(t+\tau)}$

$\alpha = 0$: Itô $\qquad \alpha = \frac{1}{2}$: Stratonovich

Using this we can compute $a_n(y)$:

$$\langle \Delta Y \rangle = \underbrace{\int_t^{t+\tau} ds \langle f(Y(s)) \rangle}_{\to\, \tau f(Y(t))} + \underbrace{\langle \int_t^{t+\tau} ds\, g(Y(s))\, \xi(s) \rangle}_{=:\, \langle \Delta Y \rangle_2}$$

$$\langle \Delta Y \rangle_2 = \langle g(Y^*) \int_t^{t+\tau} ds\, \xi(s) \rangle$$

For $\tau \to 0$ we have

$$g(Y^*) = g((1-\alpha)Y(t) + \alpha Y(t+\tau)) =$$

$$= g(Y(t) + \alpha(Y(t+\tau) - Y(t))) \simeq$$

$$\simeq g(Y(t)) + \alpha\, g'(Y(t))\, (Y(t+\tau) - Y(t))$$

$$\Rightarrow \quad \langle \Delta Y \rangle_2 = \alpha\, g'(Y(t)) \left\langle \left(Y(t+\tau) - Y(t)\right) \int_t^{t+\tau} ds\, \xi(s) \right\rangle$$

$$= \alpha\, g'(Y(t)) \left\langle g(Y^*) \underbrace{\int_t^{t+\tau} ds' \int_t^{t+\tau} ds\, \xi(s')\, \xi(s) \rangle}_{\to\, g(Y(t))\,\tau \;+\; O(\tau^2)}$$

because the difference between Y^* and $Y(t)$ leads to higher order corrections

$$\Rightarrow \quad \underline{a_1(y) = f(y) + \alpha\, g'(y)\, g(y)}$$

<u>Remarks:</u>

(i) The result obtained assuming a smooth variable transformation $Y \to \tilde{Y}$ corresponds to the Stratonovich interpretation ($\alpha = \frac{1}{2}$). This is applicable whenever $\xi(t)$ is used as a approximation of a continuous process with finite correlation time.

(ii) The Itô interpretation leads to the FP-eq

$$\frac{\partial P_1}{\partial t} = -\frac{\partial}{\partial y} f(y) P_1 + \frac{1}{2} \frac{\partial^2}{\partial y^2} g^2(y) P_1$$

It is applicable when the FP-eq. is used as an approximation for a jump process where the rates depend on the initial (rather than final) state ($\alpha = 0$).

(iii) Under the Itô interpretation the standard rules of variable transformation do not apply $\Rightarrow$ Itô calculus, stochastic integration