$\frac{1}{N} S_N \xrightarrow{N\to\infty} \mu_1$ provided μ_1 exists,

in the following precise sense:

$$\lim_{N\to\infty} \text{Prob}\left[\left| \frac{1}{N} S_N - \mu_1 \right| > \varepsilon \right] = 0 \quad \text{for any } \varepsilon > 0$$

or some similar formulation. In the following we are interested in the fluctuations of S_N around its mean.

a) Central limit theorem (CLT)

Assume for simplicity that $\mu_1 = 0$, and that $\sigma^2 < \infty$. Then the CLT states that the distribution of the RV

$$Z_N := \frac{1}{\sqrt{N}} S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i$$

becomes Gaussian for $N \to \infty$,

$$f_N(z) \xrightarrow{N\to\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-z^2/2\sigma^2}$$

Proof: $G_{Z_N}(k) = \langle e^{ikX/\sqrt{N}} \rangle_X^N = G_X(k/\sqrt{N})^N$

$$\Rightarrow \ln G_{Z_N}(k) = N \underbrace{\ln G_X(k/\sqrt{N})}_{\simeq\, -\frac{1}{2}\sigma^2 \frac{k^2}{N} + \frac{1}{6}\frac{\kappa_3}{N^{3/2}}(ik)^3 + \dots}$$

$$\Rightarrow \lim_{N\to\infty} \ln G_{Z_N}(k) = -\frac{1}{2}\sigma^2 k^2$$

and all higher cumulants vanish under this scaling. $\square$

b) Lévy laws

What happens if $\sigma^2 = \infty$, e.g. if

$$f(x) \sim |x|^{-(\alpha+1)}, \quad |x| \to \infty$$

with $\alpha < 2$?

Consider for example the Cauchy distribution

$$f(x) = \frac{c}{\pi} \frac{1}{c^2 + x^2}$$

$$\Rightarrow \quad G_x(k) = e^{-C|k|} \simeq 1 - C|k|, \quad k \to 0$$

Define now $\quad Z_N = \frac{1}{N} \sum_{i=1}^{N} X_i = \frac{1}{N} S_N$

$$\Rightarrow \quad G_{Z_N}(k) = \left(e^{-C\frac{|k|}{N}}\right)^N = e^{-C|k|}$$

$\Rightarrow$ Cauchy distribution is a stable law, which is invariant under addition & rescaling of RV's. In contrast to the CLT case,

- $G_x(k)$ is non-analytic at $k=0$, which
- requires a different scaling of S_N.

This picture generalizes to any $\alpha < 2$; assuming for now that $f(x) = f(-x)$, we have

(i) $G_x(k) \simeq 1 - C|k|^\alpha, \quad |k| \to 0$

(ii) Rescaling $\quad Z_N = \frac{1}{N^{1/\alpha}} S_N \xrightarrow{N\to\infty} Z$ with

(iii) $f(z) = L_{\alpha,0}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ikz} e^{-C|k|^\alpha}$

symmetric Lévy law.

In the general, asymmetric case one has to distinguish between $1<\alpha<2$ and $0<\alpha<1$:

$$\left.\begin{array}{ll} \underline{1<\alpha<2:} & z_N = \frac{1}{N^{1/\alpha}}\left(S_N - \mu_1 N\right) \\ \underline{0<\alpha<1:} & z_N = \frac{1}{N^{1/\alpha}} S_N \end{array}\right\}$$

converges to RV with pdf

$$L_{\alpha,\beta}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-izk}\, e^{-C|k|^{\alpha}\left(1+i\beta\,\mathrm{sgn}(k)\,\mathrm{tg}\frac{\pi\alpha}{2}\right)}$$

$$\beta = \frac{C_+ - C_-}{C_+ + C_-}\,, \quad f(x) \sim \begin{cases} C_-|x|^{-(\alpha+1)} & x\to-\infty \\ C_+|x|^{-(\alpha+1)} & x\to\infty \end{cases}$$

Explicit expressions for the limit laws exist only for $\alpha=1$, $\beta=0$ (Cauchy) and $\alpha=\frac{1}{2}$, $\beta=1$:

$$L_{\frac{1}{2},1}(z) = \begin{cases} \frac{C}{\sqrt{2\pi}}\, z^{-3/2}\, e^{-C^2/2z} & z>0 \\ 0 & z<0 \end{cases}$$

<u>Remark:</u> Power law distributions with $\alpha > 2$ display higher order singularities in $G_X(k)$.

<u>Example:</u> $$f(x) = \frac{2c^3}{\pi (x^2+c^2)^2} \qquad (\alpha = 3)$$

$$\Rightarrow G_X(k) = (1 + c|k|)\, e^{-c|k|} \simeq 1 - \frac{1}{2}c^2k^2 + \underbrace{\frac{c^3}{3}|k|^3}_{\sim |k|^\alpha} + \dots$$

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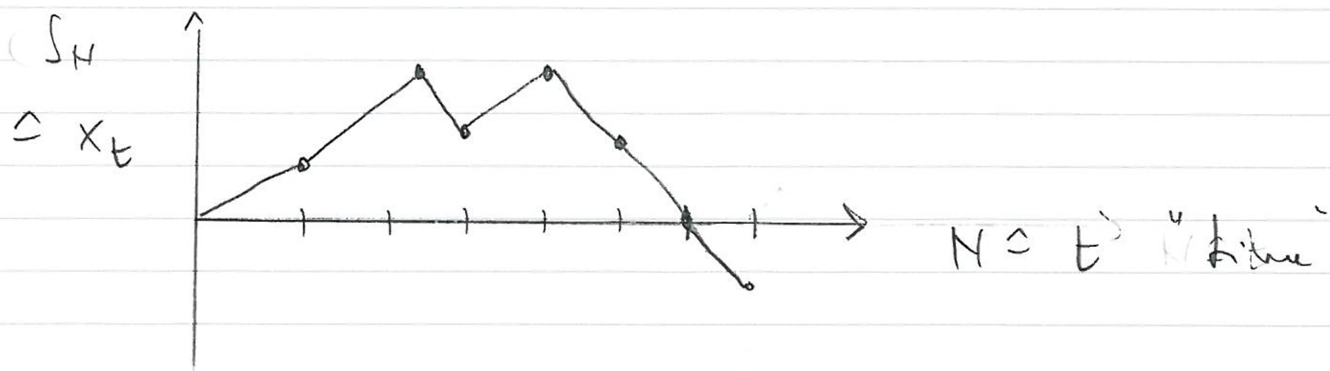
c) <u>Random walks and Lévy flights</u>

Let $S_N = \sum_{i=1}^{N} X_i$ with X_i i.i.d from a symmetric distribution. Then S_N can be interpreted as the position of a <u>one-dimensional random walk (RW)</u>:



Generalization to d dimensions:

$$\vec{R}_N = \sum_{i=1}^{N} \vec{r}_i \,, \quad \vec{r}_i = (x_1, \dots, x_d)\,, \; x_i \text{ i.i.d., symmetric.}$$

Two cases: (i) $\langle X_i^2 \rangle = \sigma^2 < \infty$

$\Rightarrow$ CLT states that $\frac{1}{\sqrt{N}} \vec{R}_N \xrightarrow[N\to\infty]{}$ Gaussian,

$$\langle |\vec{R}_N|^2 \rangle \simeq d\,\sigma^2 N \quad \text{standard diffusion}$$

(ii) $\langle x_i^2 \rangle = \infty$, $f(x) \sim x^{-(\alpha+1)}$ with $\alpha < 2$

$\Rightarrow N^{-1/\alpha} \vec{R}_N \xrightarrow[N\to\infty]{}$ Lévy law, which implies that typically

$$|\vec{R}_N| \sim N^{1/\alpha} \quad \text{superdiffusive, } \frac{1}{\alpha} > \frac{1}{2}.$$

But note that the mean square displacement does not exist.

RW's with $\alpha < 2$ are also called Lévy flights.

Examples:

(i) Two-dimensional Lorentz gas

(ii) Living polymers

(iii) Foraging behavior in animals

(iv) Stock prices

(ii) <u>Living polymers</u> (Ott et al., PRL 65, 2201, 1990)

"Living polymers" are chains of micelles that break up & recombine to produce a stationary length distribution

$$f(l) = l_0^{-2}\, l\, e^{-l/l_0}$$

A tracer particle on a polymer of length l diffuses with diffusion constant

$$D(l) = D_0 \, (l_D / l)^{\gamma} \sim l^{-\gamma}, \quad \gamma \simeq 2.15$$

$\Rightarrow$ distance covered during time τ^* between two breakup / recombination events is

$$r(l) = \sqrt{D(l)\,\tau^*} \sim l^{-\gamma/2}, \quad l \sim r^{-2/\gamma}$$

$$\Rightarrow \quad f(r) \sim \frac{dl}{dr} f(l) \sim r^{-\left(\frac{2}{\gamma}+1\right)} r^{-\frac{2}{\gamma}} \underbrace{e^{-l/l_0}}_{\to 1,\ r\to\infty}$$

$$\sim r^{-\left(\frac{4}{\gamma}+1\right)}$$

$\Rightarrow$ Lévy-flight with $\alpha = \frac{4}{\gamma} \simeq 1.86 < 2.$

(iii) Foraging behaviour of animals

Viswanathan et al., Nature 381, 413 (1996)
find a power law distribution of flight
times of wandering albatross with $\alpha \simeq 1$.
This finding was subsequently revisited & questioned
by the same group (Edwards et al.,
Nature 449, 1044, 2007).

(iv) Stock prices

Let $S_N = \sum_{i=1}^{N} X_i$ be a detrended stock price,
X_i price increments at high temporal resolution.
Then the distribution of the X_i is a
truncated power law

$$f(x) = C\,|x|^{-(\alpha+1)}\, e^{-|x|/\xi}, \quad \alpha \simeq 1.4$$

(Mantegna, Stanley, Nature 376, 46, 1995)

This implies a crossover in S_N from Lévy-
behaviour at short times to Gaussian behaviour at
long times.

Estimate of the crossover time:

- variance of X is

$$\sigma^2 = \langle x^2 \rangle = \int dx\, |x|^{(1-\alpha)} e^{-|x|/\xi} \sim \int^{\xi} dx\, x^{1-\alpha} \sim \xi^{(2-\alpha)}$$

- Thus

$$S_N \sim \begin{cases} N^{1/\alpha} & \text{short times} \\ \sqrt{\xi^{(2-\alpha)} N} & \text{long times} \end{cases}$$

$\Rightarrow$ crossover at $\xi^{(2-\alpha)} N \sim N^{2/\alpha} \Rightarrow \underline{N_x \sim \xi^{\alpha}}$

Detailed analysis: I. Koponen, PRE $\underline{52}$, 1197, 1995

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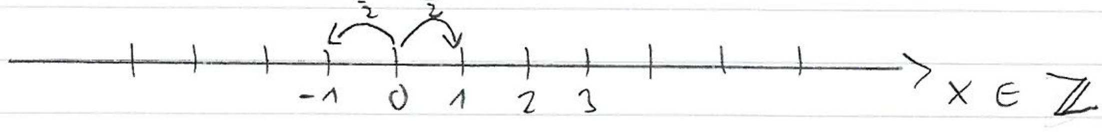
d) Subdiffusion and continuous time RW's

Consider RW on a (one-dim.) lattice:



$\Rightarrow$ displacement after n steps is $\langle x_n^2 \rangle = n$.

In the continuous time RW (CTRW) the walker waits a random waiting time τ

between two jumps, with pdf $\psi(\tau)$.

Two cases:

(i) $\langle \tau \rangle < \infty \Rightarrow \langle x(t)^2 \rangle \simeq \frac{t}{\langle \tau \rangle}$

normal diffusion

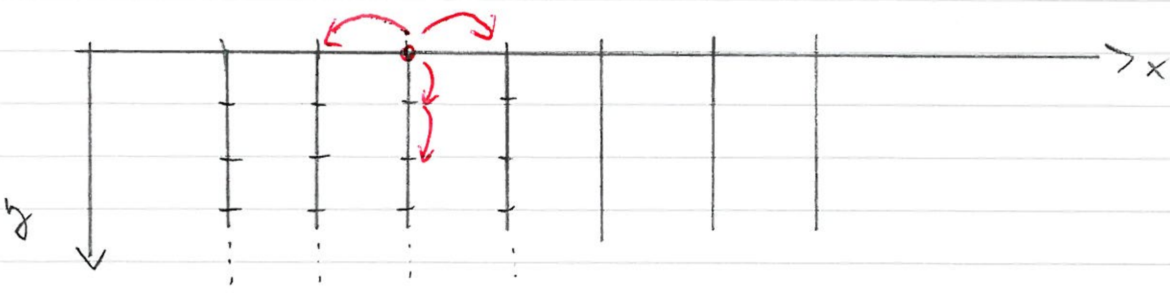
(ii) $\langle \tau \rangle = \infty$, e.g. $\psi(\tau) \sim \tau^{-(\alpha+1)}$ with $\alpha < 1$

Then the time elapsed after n hops is a Lévy variable

$$t_n = \sum_{i=1}^{n} \tau_i \sim n^{1/\alpha} \gg n$$

$$\Rightarrow \quad x_n^2 \sim n \sim t^{\alpha} \ll t \qquad \text{subdiffusion}$$

Example: Diffusion in combs (Havlin, Ben-Avraham 1987)



pdf of return times to $y=0$ is $\psi(\tau) \sim \tau^{-3/2}$

$$\Rightarrow \quad x_t^2 \sim t^{1/2}$$

In the most general CTRW, jump lengths are drawn from pdf $f(x)$ and waiting times from pdf $\psi(\tau)$. Then if

$$\left.\begin{array}{l} f(x) \sim x^{-(\alpha_x+1)}, \quad \psi(\tau) \sim \tau^{-(\alpha_t+1)} \\ \alpha_x < 2, \quad \alpha_t < 1 \end{array}\right\}$$

we can estimate

$$\left.\begin{array}{l} x_t^2 \sim n^{1/\alpha_x}, \quad n \sim t^{\alpha_t} \\ \Rightarrow \underline{x_t^2 \sim n^{\alpha_t/\alpha_x}} \end{array}\right\}$$

but the computation of the distribution of x is quite non trivial.

3° Large deviations

Motivation: Consider binomial RV

$$X = \begin{cases} 1 & \text{prob. } p \quad \text{"success"} \\ 0 & \text{prob } 1-p \quad \text{"failure"} \end{cases}$$

$$\Rightarrow \mu_1 = \frac{1}{2}, \quad \sigma^2 = p(1-p)$$