b) The limit theorem (Fisher and Tippett, 1928; Gnedenko, 1943)

Suppose there is a sequence of constants a_N, b_N with $a_N > 0$, such that the RV

$$Y_N = \frac{X_{max}(N) - b_N}{a_N}$$

has a non-degenerate distribution for $N \to \infty$, i.e. that the limit

$$\lim_{N \to \infty} F(a_N y + b_N)^N = G(y)$$

exists and is non-degenerate. Then G belongs to the one-parameter family

$$G_\gamma(y) = \exp\left(-(1+\gamma y)^{-1/\gamma}\right), \quad 1+\gamma y > 0$$

with $\gamma \in \mathbb{R}$. In the case $\gamma = 0$

$$G_0(y) = \exp(-e^{-x})$$

γ is called the extreme value index.
In the explicit examples we found that

(i) exponential: $b_N = \ln N$, $a_N = 1$, $\gamma = 0$

(ii) uniform: $b_N = 1 - \frac{1}{N}$, $a_N = \frac{1}{N}$, $\gamma = -1$

(iii) Pareto: $b_N = 0$, $a_N = N^{1/\alpha}$, $\gamma = 1/\alpha$

In general one distinguishes three classes of extreme value distributions:

$\gamma > 0$: Fréchet, unbounded with power law tail

$\gamma = 0$: Gumbel, unbounded[1] with decay faster than a power law[1] or bounded

$\gamma < 0$: Weibull, bounded support with power law behavior at the boundary.

We derive G_γ using a renormalization approach (Bertin, Györgyi, 2010).

The combination of maximization, shift and rescaling defines an RG transformation on F,

$$[\hat{R}_N F](x) = F^N(a_N x + b_N) \qquad \text{RG}$$

Setting $N = p^n$ we can perform the transformation iteratively by applying $\hat{R}_p$ n times, $\hat{R}_N = (\hat{R}_p)^n$.
This suggests to use a "time scale" s that is linear in n by defining $s = \ln N$ and

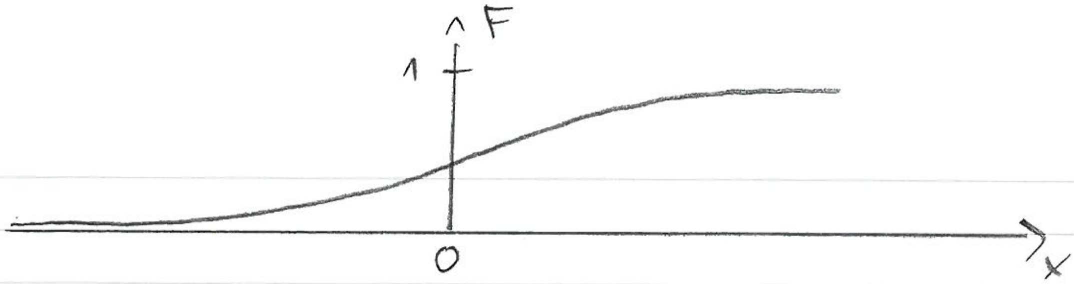
$$F(x,s) = [\hat{R}_{e^s} F](x), \qquad F(x,0) = F(x)$$

abuse of notation! "parent distribution"

Moreover, for convenience we write

$$F(x,s) = \exp(-e^{-g(x,s)}), \quad F(x) = \exp(-e^{-g(x)})$$

Similarly $a_N = a_{e^s} = a(s)$, $b_N = b(s)$.
We fix $a(s)$, $b(s)$ through the requirement that RG tends to a non-degenerate limit for $N \to \infty$:



$\Rightarrow$ require that $F(0,s) = 1/e, \ \partial_x F(0,s) = 1/e \quad \forall s.$
(or some other number)

With $\partial_x F = \partial_x g \cdot e^{-g-e^{-g}}$ this implies

$$g(0,s) = 0 \ , \ \partial_x g(0,s) = 1 \qquad (1)$$

In this notation RG reads

$$F(x,s) = F(a(s)x + b(s))^{e^s}$$

$$\Rightarrow g(x,s) = -\ln(-\ln F(x,s)) = -\ln(e^s(-\ln F(ax+b)))$$
$$= -s - \ln(-\ln F(ax+b))$$
$$= g(a(s)x + b(s)) - s \qquad (2)$$

To determine $a(s)$ and $b(s)$ we insert (2) in (1):

$$g(0,s) = g(b(s)) - s = 0 \quad \Rightarrow \quad b(s) = g^{-1}(s) \qquad (3)$$

$$\partial_x g(x,s) = a(s)\, g'(a(s)x + b(s)) \overset{!}{=} 1 \quad \text{at} \quad x=0$$

$$\Rightarrow \quad a(s) = 1/g'(b(s)) \qquad (4)$$

To derive a flow equation for $g(x,s)$ we take the derivative of (2) w.r.t. s:

$$\partial_s g(x,s) = (\dot{a}x + \dot{b})\, g'(ax+b) - 1$$

To determine $\dot{a}, \dot{b}$ we use (3), (4):

$$\left.\begin{aligned} &\frac{d}{ds} g(b(s)) = 1 \quad \Rightarrow \quad \dot{b} = 1/g'(b(s)) = a \\ &\dot{a} = \frac{d}{ds}\left(1/g'(b(s))\right) = -\frac{1}{g'(b(s))^2}\, g''(b(s))\, \dot{b} \end{aligned}\right\}$$

$$\Rightarrow \quad \frac{\dot{a}}{a} = -\frac{g''(b(s))}{g'(b(s))^2} := \gamma(s)$$

Moreover from $\partial_x g(x,s) = 1$ we have

$$g'(ax+b) = \frac{1}{a}\, \partial_x g(x,s)$$

$$\Rightarrow \quad \underline{\partial_s g(x,s)} = \frac{\dot{a}x + \dot{b}}{a}\, \partial_x g(x,s) - 1 =$$

$$= \underline{(1+\gamma x)\, \partial_x g(x,s) - 1}$$

an autonomous eq. for $g(x,s)$, with all information about the parent distribution contained in $\gamma(s)$.

<u>Fixed point:</u> $\gamma(s) = \gamma = \text{const.}, \; \partial_s g = 0$

$$\Rightarrow \quad (1+\gamma x)\frac{d}{dx} g^*(x) = 1 \;, \quad g^*(0) = 0$$

$$\Rightarrow \quad g^*(x) = \int_0^x \frac{dx'}{1+\gamma x'} = \frac{1}{\gamma}\ln(1+\gamma x)$$

Transforming back to the original distribution we have

$$F^*(x) = e^{-e^{-g^*(x)}} = e^{-(1+\gamma x)^{-1/\gamma}} = G_\gamma(x) \quad \square$$

<u>Examples:</u>

(i) <u>exponential:</u> $F(x) = 1 - e^{-x}$

$$\Rightarrow \quad g(x) = -\ln(-\ln F(x)) = -\ln(\underbrace{-\ln(1-e^{-x})}_{\simeq e^{-x}})) \simeq x$$

$\Rightarrow \; g'' = 0, \; \underline{\gamma = 0}$. The scale parameters are

$$b(s) = g^{-1}(s) = s = \ln N \;, \quad a(s) = \dot{b} = 1.$$

(ii) <u>Pareto:</u> $F(x) = 1 - x^{-\alpha} \Rightarrow g(x) \simeq \alpha \ln x$

$$\Rightarrow \quad g' = \frac{\alpha}{x}, \; g'' = -\frac{\alpha}{x^2} \quad \Rightarrow \underline{\gamma = -\frac{g''}{(g')^2} = 1/\alpha}$$

(iii) uniform: $F(x) = x$, $g(x) = -\ln\ln(1/x)$
$b(s) = e^{-e^{-s}} \to 1$, $s \to \infty$

$$\Rightarrow g'(x) = \frac{1}{x\ln(1/x)}, \quad g'' = -\frac{1}{x^2 \ln(1/x)} + \frac{1}{x^2 \ln(1/x)^2}$$

$$\Rightarrow \gamma = -\left.\frac{g''}{g'^2}\right|_{x=1} = -1$$

(iv) Gaussian: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$\Rightarrow 1 - F(x) \simeq \frac{1}{x}\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \to \infty$$

$$\Rightarrow g(x) = -\ln(-\ln F) \simeq \frac{x^2}{2} + \ln x + \frac{1}{2}\ln(2\pi)$$

$$g' = x + \frac{1}{x}, \quad g'' = 1 - \frac{1}{x^2}$$

$$\Rightarrow \gamma = -\frac{g''}{(g')^2} \simeq -\frac{1}{x^2} \to 0, \quad x \to \infty.$$

Gumbel class

The scale parameter $b(s)$ is the solution of

$$g(b) = \frac{b^2}{2} + \ln b + \frac{1}{2}\ln(2\pi) = s \Rightarrow b \simeq \sqrt{2s}$$

to leading order.

$\Rightarrow a(s) = \dot{b} = 1/\sqrt{2s} \to 0, \; s \to \infty.$

In the original variables this implies

$$y_N = \frac{x_{max}(N) - \sqrt{2 \ln N}}{1/\sqrt{2 \ln N}}$$

$\Rightarrow$ typical value $\sqrt{2 \ln N}$ [1] with fluctuation $\sim 1/\sqrt{2 \ln N}$, Gumbel distributed.

[1] more precise: $(2 \ln N - \ln \ln N - \ln 4\pi)^{1/2} = b_N$

c) Application: Random energy model

N spins, 2^N configurations with energies drawn randomly from Gaussian distribution

$$f(E) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-E^2/2\sigma^2}, \quad \langle E \rangle = 0.$$

Then the expected value of the ground state is determined by

$$\frac{E_{min}(N) + \sqrt{2\sigma^2 \ln 2^N}}{1/\sqrt{2\sigma^2 \ln 2^N}} = \gamma_E$$

mean of Gumbel distr.

but note that there are corrections to b_N

$$\Rightarrow \quad E_{min}(N) \simeq -\sqrt{2 \ln 2\, \sigma^2 N}$$

In order for the ground state energy to be extensive we need $\sigma^2 \sim N \Rightarrow$ set $\sigma^2 = N$.
Then we know that

$$\lim_{\beta \to \infty} \left(-\frac{1}{\beta} \langle \ln Z_N(\beta) \rangle \right) = E_{min}(N) = -\sqrt{2 \ln 2}\, N$$

For comparison we compute the "annealed" free energy

$$\mathcal{F}_N^{an}(\beta) = -\frac{1}{\beta} \ln \langle Z_N(\beta) \rangle$$

We have

$$\langle Z_N(\beta) \rangle = \left\langle \sum_{i=1}^{2^N} e^{-\beta E_i} \right\rangle = 2^N \langle e^{-\beta E_i} \rangle$$

$$\langle e^{-\beta E_i} \rangle = \int_{-\infty}^{\infty} dE \, \frac{1}{\sqrt{2\pi N}} e^{-E^2/2N} e^{-\beta E}$$

$$= e^{\frac{N}{2} \beta^2}$$

$$\Rightarrow \quad \mathcal{F}_N(\beta) = -\frac{1}{\beta} \left(N \ln 2 + \frac{N}{2} \beta^2 \right) =$$

$$= -N\left(\frac{\ln 2}{\beta} + \frac{\beta}{2}\right) = -N\left(\ln 2 \cdot T + \frac{1}{2T}\right) \qquad (k_B = 1)$$

$\Rightarrow$ $\mathcal{F}_N^{an}(\beta) \to -\infty$ for $\beta \to \infty$, whereas the true free energy cannot be smaller than E_{min}.

We can also compute the annealed entropy:

$$S_N^{an} = -\frac{\partial \mathcal{F}_N^{an}}{\partial T} = N \ln 2 - \frac{N}{2T^2} = 0$$

at $T_c = 1/\sqrt{2 \ln 2}$. At this temperature the annealed free energy is

$$\mathcal{F}_N^{a}(T_c) = -N\sqrt{2\ln 2} = E_{min}(N)$$

$\Rightarrow$ at $T = T_c$ the system freezes into the ground state

$0 \longrightarrow T_c \longrightarrow T$

$\mathcal{F}_N^{an} \neq \mathcal{F}_N$	$\mathcal{F}_N^{an} \simeq \mathcal{F}_N$
$\mathcal{F}_N = E_{min}$	$S_N > 0$
$S_N = 0$	