d) Counting statistics for renewal processes

For a general stationary point process we may consider the RV

N_T = number of events in a time interval of length T

For independent events we know that N_T is Poisson distributed, in particular

$$\mathrm{Var}(N_T) = \langle N_T^2\rangle - \langle N_T\rangle^2 = \langle N_T\rangle = \rho T$$

As a measure of fluctuations relative to the independent case define the index of dispersion or Fano factor

$$I_T := \frac{\langle N_T^2\rangle - \langle N_T\rangle^2}{\langle N_T\rangle}$$

- Under which conditions is $\lim_{T\to\infty} I_T = I$ finite and non zero?
- When is $I_T > 1$ (super-Poisson) or $I_T < 1$ (sub-Poisson)?

We answer these questions for a general renewal process with waiting time distribution $f_\Delta(t)$ which satisfies the CLT. Then the time T_n of the nth event

$$T_n = \sum_{i=1}^{n} t_i$$

is distributed according to

$$f_n(t) = \frac{1}{\sqrt{2\pi\sigma_\Delta^2 n}} \exp\left[-\frac{(t-\mu_\Delta n)^2}{2n\sigma_\Delta^2}\right], \quad n \to \infty$$

where μ_Δ and σ_Δ denote the mean and variance of the waiting time. For large n we have

$$T_n/n \to \mu_\Delta + O(1/\sqrt{n})$$

which also implies that

$$N_T/T \to \frac{1}{\mu_\Delta} + O(1/T^{1/2})$$

We can therefore derive the pdf $f_T(N)$ of N_T from $f_n(t)$ through a deterministic change of variables:

$$N_T \simeq t_n / \mu_\Delta \quad = \quad \frac{dN_T}{dt_n} = \frac{1}{\mu_\Delta}$$

$$\Rightarrow f_T(N) = \frac{dt_n}{dN_T} f_n(t) = \quad (*)$$

$$= \mu_\Delta \cdot \frac{1}{\left[2\pi\sigma_\Delta^2 (T/\mu_\Delta)\right]^{1/2}} \exp\left[- \frac{(N\mu_\Delta - T)^2}{2\sigma_\Delta^2 (T/\mu_\Delta)} \right] =$$

$$= \frac{1}{\left[2\pi\left(\frac{\sigma_\Delta^2}{\mu_\Delta^3}\right)T\right]^{1/2}} \exp\left[- \frac{(N - T/\mu_\Delta)^2}{2\left(\frac{\sigma_\Delta^2}{\mu_\Delta^3}\right)T} \right]$$

$\Rightarrow N$ is Gaussian with mean $\langle N \rangle = T/\mu_\Delta$ and variance

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{\sigma_\Delta^2}{\mu_\Delta^3} T = \frac{\sigma_\Delta^2}{\mu_\Delta^2} \langle N \rangle$$

$$\Rightarrow F_T = \frac{\sigma_\Delta^2}{\mu_\Delta^2} = CV(\Delta)^2 \quad \text{for } T \to \infty$$

$CV(\Delta) = \sigma_\Delta / \mu_\Delta$ is called the coefficient of variation of Δ. For the exponential distribution $CV(\Delta) = 1$.

Examples: (i) Pareto: $f_\Delta(t) = \alpha(1+t)^{-(\alpha+1)}$, $t>0$

has finite variance for $\alpha > 2$.

Then $\mu_\Delta = \frac{1}{\alpha-1}$, $\langle \Delta^2 \rangle = \frac{2}{(\alpha-1)(\alpha-2)}$

$$\Rightarrow I_\alpha = \frac{\langle \Delta^2 \rangle}{\mu_\Delta^2} - 1 = \frac{\alpha}{\alpha-2} \underline{> 1}$$

$\Rightarrow$ super-Poissonian fluctuations.

(ii) Generalized exponential: $f_\Delta(t) = N_\beta \, e^{-t^\beta}$, $\beta>0$

$$\Rightarrow I_\beta \begin{cases} < 1 & \beta > 1 \text{ "squeezed"} \\ > 1 & \beta < 1 \text{ "stretched"} \end{cases} \text{ exponential}$$

In particular, for a semi-Gaussian distribution

$$f_\Delta(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$$

one finds $I_{1/2} = \frac{\sqrt{\pi}}{2} - 1 \simeq 0.772$.

Problems

III. Stochastic processes

1° Basic concepts

a) Definitions

A stochastic process is

(i) a RV X with pdf $f(x)$

(ii) a mapping $X \longrightarrow Y_X(t)$, $t \in \mathbb{R}$

$Y_X(t)$ is called a realization of the process.

Two complementary points of view:

- for given X, $Y_X(t)$ is a function of t
- for given t, $Y_X(t)$ is a RV with pdf

$$P_1(y,t) = \langle \delta(y - Y_X(t)) \rangle_X$$

Examples:

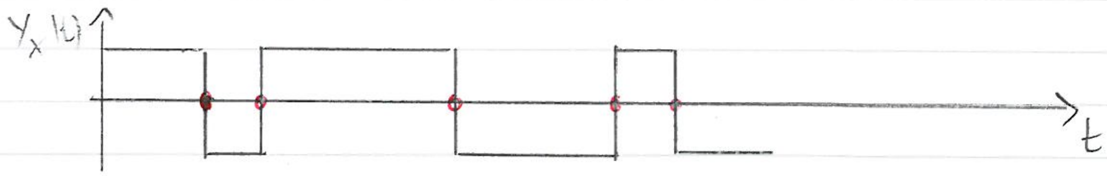
(i) $X = (A, \omega, \phi) \in \mathbb{R}^3$

$$Y_X(t) = \underbrace{A}_{\text{amplitude}} \sin(\underbrace{\omega}_{\text{frequency}} t + \underbrace{\phi}_{\text{phase}})$$

(ii) $X = \{ s; \tau_1, \dots, \tau_s \}$ a point process (Ch II)
$Y_X(t) = N_{[0,t]}$ the corresponding counting process.
In particular, when X is shot noise, Y_X is the Poisson process.

(iii) Telegraph process: X = shot noise,

$$Y_X(t) = \begin{cases} 1 & N_{[0,t]} \text{ even} \\ -1 & N_{[0,t]} \text{ odd} \end{cases}$$



(iv) Continuous time random walks: (see I.2°)

$X = \{ x_i, \tau_i \}$

increments waiting times

$$\left.\begin{aligned} \Rightarrow \quad Y_X(t) &= \sum_{i=1}^{N_{[0,t]}} x_i \\ N_{[0,t]} &= \max \{ n : \sum_{i=1}^{n} \tau_i < t \} \end{aligned}\right\}$$

b) Moments and correlations

In analogy to simple RV's (Ch I) we define

omit subscript x

$$\begin{cases} \langle Y(t)\rangle := \langle Y_x(t)\rangle_x = \int dx\, f(x)\, Y_x(t) \\ \langle Y(t_1)Y(t_2)\ldots Y(t_n)\rangle := \int dx\, f(x)\, Y_x(t_1)\ldots Y_x(t_n) \end{cases}$$

Autocorrelation function:

$K(t_1, t_2) := \mathrm{Cov}(Y(t_1), Y(t_2)) =$

$= \langle Y(t_1)Y(t_2)\rangle - \langle Y(t_1)\rangle\langle Y(t_2)\rangle$

and in the n-dimensional case:

$\vec{Y}(t) = (Y_1(t), \ldots, Y_n(t)) \in \mathbb{R}^n$

$\Rightarrow$ define $K_{ij}(t_1, t_2) = \mathrm{Cov}(Y_i(t_1), Y_2(t_2))$

$$\left.\begin{array}{l} i = j: \text{ autocorrelation function} \\ i \neq j: \text{ cross-correlation} \end{array}\right\}$$

In analogy to the generating function $G_x(k)$ we define the generating functional acting on test functions $k(t)$:

$$G[k(t)] = \left\langle \exp\left(i \int_{-\infty}^{\infty} dt \, k(t) Y(t) \right) \right\rangle$$

with the expansion

$$G[k(t)] = \sum_{m=0}^{\infty} \frac{i^m}{m!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_m \, k(t_1) \ldots k(t_m) \times$$
$$\times \langle Y(t_1) \ldots Y(t_m) \rangle$$

$$\ln G[k(t)] = \sum_{m=1}^{\infty} \frac{i^m}{m!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_m \, k(t_1) \ldots k(t_m) \times$$

$$\times \underbrace{\langle Y(t_1) \ldots Y(t_m) \rangle_c}_{\text{cumulant}}$$

In particular, $K(t_1, t_2) = \langle Y(t_1) Y(t_2) \rangle_c$.

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c) <u>Stationarity</u> (see II. 1°)

A stochastic process is called stationary if

$$\langle Y(t_1 + \tau) \ldots Y(t_m + \tau) \rangle = \langle Y(t_1) \ldots Y(t_m) \rangle$$

for all τ, m, $t_1 \ldots, t_m$. This implies in particular that

$$\left.\begin{array}{l} \langle Y(t+\tau)\rangle = \langle Y(t)\rangle = \bar{Y} = \text{const.} \\ \kappa(t_1, t_2) = \kappa(t_1 - t_2) = \kappa(t_1 - t_2) \end{array}\right\}$$

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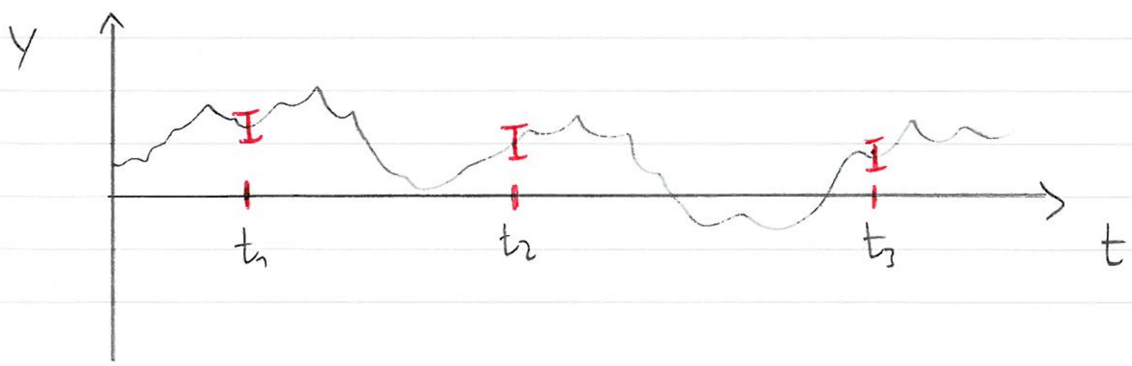
d) <u>Distribution functions</u>

A general stochastic process can be fully characterized through the hierarchy of distribution functions defined by:

$$P_n(y_1, t_1; y_2, t_2; \dots; y_n, t_n) =$$

$$= \langle \prod_{i=1}^{n} \delta(y_i - Y_X(t_i)) \rangle_X$$

$P_n\, dy_1 \dots dy_n$ is the <u>joint probability</u> for the process to be in $(y_1, y_1 + dy_1)$ at t_1, $(y_2, y_2 + dy_2)$ at t_2 etc.:



Similarly conditional probabilities can be defined:

$$P_{1|1}(y_2, t_2 | y_1, t_1) = \frac{P_2(y_1, t_1; y_2, t_2)}{P_1(y_1, t_1)}$$

and generally

$$P_{l|k}(y_{k+1}, t_{k+1}; \dots; y_{k+l}, t_{k+l} | y_1, t_1; \dots; y_k, t_k) =$$

$$= \frac{P_{k+l}(y_1, t_1; \dots; y_{k+l}, t_{k+l})}{P_k(y_1, t_1; \dots; y_k, t_k)}$$

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e) Gaussian processes

A stochastic process is called Gaussian, if all distribution functions $P_n(y_1, t_1; \dots; y_n, t_n)$ are $2n$-dimensional Gaussian probability density functions (cf. I. 1° f)). Then all cumulants of order ≥ 3 vanish and the generating functional (and hence the full process) is specified by $\langle Y(t)\rangle$ and $\kappa(t_1, t_2)$:

$$G[k(t)] = \exp\left[i \int_{-\infty}^{\infty} k(t_1)\, \langle Y(t_1)\rangle \, dt_1 - \right.$$

$$\left. - \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; k(t_1)\, k(t_2)\, \kappa(t_1, t_2) \right]$$

If the process is also stationary, it is fully specified (up to a constant) by the auto-correlation function $\kappa(\tau)$.