is specified by $\langle Y(t)\rangle$ and $K(t_1,t_2)$:

$$G[k(t)] = \exp\Big[i \int_{-\infty}^{\infty} k(t_1)\,\langle Y(t_1)\rangle\, dt_1 - \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, k(t_1)\,k(t_2)\,K(t_1,t_2)\Big]$$

If the process is also stationary, it is fully specified (up to a constant) by the auto-correlation function $K(\tau)$.

2° Spectral analysis and 1/f-noise

a) Wiener-Khinchin-Theorem

Consider stationary stochastic process $Y(t)$ with $\langle Y\rangle = 0$, $\langle Y^2\rangle = \sigma^2$ on a time interval of length T. Then we can decompose $Y(t)$ into Fourier components

$$
\left.\begin{aligned}
Y(t) &= \sum_{n=1}^{\infty} A_n \sin(\omega_n t), \qquad \omega_n = \frac{n\pi}{T} \\
A_n &= \frac{2}{T} \int_0^T dt\, Y(t) \sin(\omega_n t) \qquad \text{RV's}
\end{aligned}\right\}
$$

Parseval - identity for the Fourier transform implies

$$
\frac{1}{T} \int_0^T dt\, Y(t)^2 \simeq \langle Y^2 \rangle = \frac{1}{2} \sum_{n=1}^{\infty} \langle A_n^2 \rangle
$$

decomposes σ^2 into contributions from different frequencies ω_n. The power spectrum $S(\omega)$ of $Y(t)$ is defined by

$$
\frac{1}{2} \sum_{\substack{n \\ \omega < \frac{\pi n}{T} < \omega + \Delta\omega}} \langle A_n^2 \rangle = S(\omega)\, \Delta\omega \qquad \left\{ \begin{array}{l} \Delta\omega \to 0 \\ T \to \infty \\ \Delta\omega \gg \pi/T \end{array} \right.
$$

According to the Wiener-Khinchin - theorem

$$
S(\omega) = \frac{2}{\pi} \int_0^\infty d\tau\, \cos(\omega\tau)\, K(\tau)
$$

<u>Proof:</u> $$\langle A_n^2 \rangle = \frac{4}{T^2} \int_0^T dt \int_0^T dt' \, \sin(\omega_n t) \sin(\omega_n t') \, K(t-t')$$

$$= \frac{4}{T^2} \int_0^T dt \, \sin(\omega_n t) \int_{-t}^{T-t} d\tau \, \sin(\omega_n (t+\tau)) K(\tau) =$$

$$= \frac{4}{T^2} \int_0^T dt \, \sin^2(\omega_n t) \int_{-t}^{T-t} d\tau \, \cos(\omega_n \tau) \, K(\tau) \; +$$

$$+ \frac{4}{T^2} \int_0^T dt \, \sin(\omega_n t) \cos(\omega_n t) \int_{-t}^{T-t} d\tau \, \sin(\omega_n \tau) \, K(\tau) =$$

$$[s = t/T]$$

$$= \frac{4}{T} \Big\{ \int_0^1 ds \, \sin^2(n\pi s) \int_{-Ts}^{(1-s)T} d\tau \, \cos(\omega_n \tau) \, K(\tau) \; +$$

$$+ \int_0^1 ds \, \sin(n\pi s) \cos(n\pi s) \int_{-Ts}^{(1-s)T} d\tau \, \sin(\omega_n \tau) K(\tau) \Big\}$$

Provided $K(\tau)$ decays sufficiently fast at long times we have, for $T \to \infty$

$$\int_{-Ts}^{(1-s)T} d\tau \cos(\omega_n \tau) K(\tau) \to 2 \int_0^\infty d\tau \cos(\omega_n \tau) K(\tau)$$

$$\int_{-Ts}^{(1-s)T} d\tau \sin(\omega_n \tau) K(\tau) \to \int_{-\infty}^{\infty} d\tau \sin(\omega_n \tau) K(\tau) = 0$$

$$\Rightarrow \quad \langle A_n^2 \rangle \to \frac{4}{T} \int_0^\infty d\tau \cos(\omega_n \tau) K(\tau)$$

and since the number of frequencies in an interval of length $\Delta\omega$ is $\frac{T}{\pi}\Delta\omega$ we have

$$\frac{1}{2} \sum_{\Delta\omega} \langle A_n^2 \rangle = \frac{1}{2} \cdot \frac{T}{\pi} \Delta\omega \cdot \frac{4}{T} \int_0^\infty d\tau \cos(\omega \tau) K(\tau)$$

$$= \frac{2}{\pi} \Delta\omega \int_0^\infty d\tau \cos(\omega\tau) K(\tau)$$

$\square$

Inverse: $\quad K(\tau) = \int_0^\infty d\omega \, S(\omega) \cos(\omega\tau)$.

Examples: (i) White noise implies that all frequencies appear with the same weight:

$$\left.\begin{aligned} S(\omega) &\equiv S_0 \\ K(\tau) &= \pi S_0 \delta(\tau) \end{aligned}\right\}$$

(ii) Lorentz-spectrum $S(\omega) = \dfrac{S_0}{1+\omega^2\tau_c^2}$

$$\Rightarrow \quad S(\omega) \approx \begin{cases} S_0 & \omega \ll \tau_c^{-1} \quad \text{effectively white} \\ S_0/\omega^2\tau_c^2 & \omega \gg \tau_c^{-1} \end{cases}$$

$$\Rightarrow \quad K(\tau) = \underbrace{\frac{\pi}{2}\left(\frac{S_0}{\tau_c}\right)}_{K(0)} e^{-|\tau|/\tau_c}$$

τ_c is the correlation time of the process.

---

b) 1/f-noise

Many natural and man-made processes display a power spectrum of the form

$$S(\omega) \simeq S_0 \, \omega^{-\beta} \quad , \quad \beta \simeq 0.8 - 1.4$$

over a large range of frequencies, $\omega_1 < \omega < \omega_2$.

Examples:
- Voltage fluctuations in resistors
- quasars, geophysics, physiology
- music ....

$\Rightarrow$ 1/f-noise is typical of "complex dynamics."

Questions: (i) Why is $\beta \simeq 1$ special?
(ii) What are the underlying physical mechanism?

Ad (i) : The variance of the process is

$$\sigma^2 = \langle Y^2 \rangle = K(0) = \int_0^\infty d\omega \, S(\omega) = S_0 \int_{\omega_1}^{\omega_2} d\omega \, \omega^{-\beta}$$

$\Rightarrow$ stationarity ($\sigma^2 < \infty$) requires cutoff frequencies:

- $\beta > 1 \Rightarrow \omega_1 > 0$ required, $\omega_2 \to \infty$ possible
- $\beta < 1 \Rightarrow \omega_2 > 0$ required, $\omega_1 \to 0$ possible

- $\beta = 1 \Rightarrow$ both upper and lower cutoffs are required.

For $\beta > 1$ and $\beta < 1$ the exponent β describes different parts of the autocorrelation function:

$\beta > 1$: Unlimited power law for $\omega \to \infty$, β describes the behavior at short times $\tau \to 0$

Analogy: $K(\tau) = \int_0^\infty d\omega\, S(\omega)\, e^{i\omega\tau} = \int_{\omega_1}^\infty d\omega\, \omega^{-\beta} e^{i\omega\tau}$

$$\downarrow$$

$$G_x(k) = \int dx\, f(x)\, e^{ikx} = \int_1^\infty dx\, \alpha x^{-(\alpha+1)} e^{ikx}$$

generating fct. for Pareto distribution with $\alpha = \beta - 1$

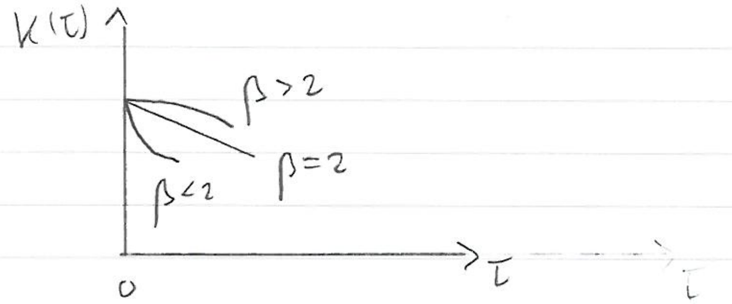
$\Rightarrow$ we know from discussion of Lévy RV's that

$$K(0) - K(\tau) \sim |\tau|^{\alpha} = |\tau|^{\beta-1}, \quad \tau \to 0$$

$\Rightarrow$ for $1 < \beta < 3$, $K(\tau)$ is singular at $\tau = 0$.

Example: Lorentz spectrum ($\beta = 2$) corresponds to Cauchy distribution ($\alpha = 1$)

For $\beta > 3$, $K(\tau)$ is smooth at $\tau = 0$



$\underline{\beta < 1:}$ Unlimited power law for $\omega \to 0$,
β describes behavior at long times $\tau \to \infty$.

$$K(\tau) = S_0 \int_0^{\omega_2} d\omega\, \omega^{-\beta} \underbrace{\cos(\omega\tau)}_{\text{rapidly oscillating for } \omega\tau \gg 1}$$

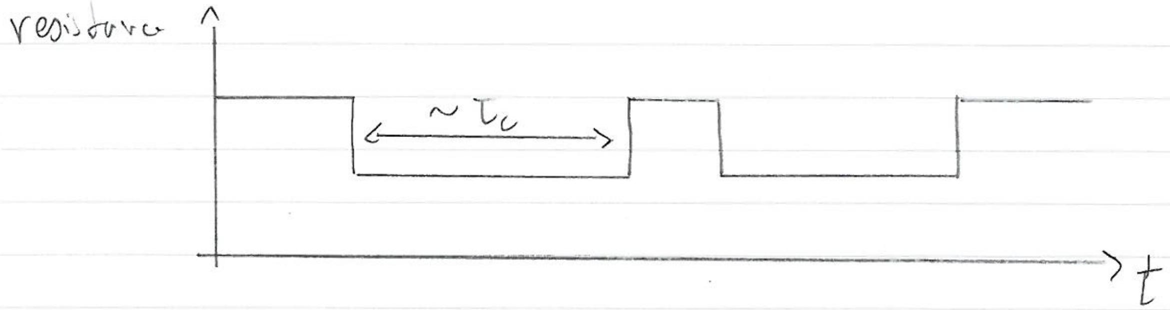
$$\simeq S_0 \int_0^{1/\tau} d\omega\, \omega^{-\beta} = \underline{\frac{S_0}{1-\beta}\, \tau^{-(1-\beta)}}$$

$\Rightarrow$ power law decay of correlations.

This type of behavior is observed in climate data ($\to$ Bunde et al., $\beta \simeq 0.3$)

Note that none of these analyses work in the borderline case $\beta = 1$.

Ad (iii): A simple model for 1/f-noise in resistors assumes the existence of defects which switch between two configurations according to a telegraph process with switching time τ_c:

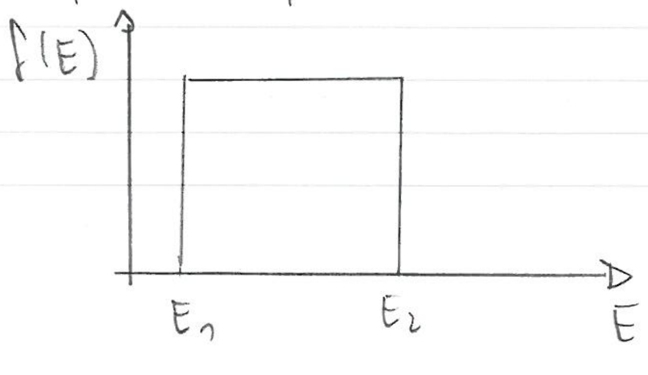


$\Rightarrow$ single defect has Lorentz spectrum → Problem

$$S(\omega) = \frac{2}{\pi} \sigma^2 \frac{\tau_c}{1+\omega^2 \tau_c^2}$$

Switching is thermally activated: $\tau_c = \tau_0 e^{E/k_B T}$
with defect energies E distributed according to a pdf $f(E)$.

Simple example: $f(E)$ is uniform in $[E_1, E_2]$



$$\Delta E = E_2 - E_1 \gg k_B T$$

Then the averaged power spectrum is

$$\bar{S}(\omega) = \frac{2}{\pi} \sigma^2 \frac{1}{\Delta E} \int_{E_1}^{E_2} dE \, \frac{\tau_0 \, e^{E/k_B T}}{1 + \omega^2 \tau_0^2 \, e^{2E/k_B T}}$$

substitute: $\tau = \tau_0 \, e^{E/k_B T}$, $d\tau = \frac{1}{k_B T} \tau \, dE$

$$= \frac{2}{\pi} \sigma^2 \frac{k_B T}{\Delta E} \int_{\tau_1}^{\tau_2} \frac{d\tau}{1 + \omega^2 \tau^2} =$$

$$= \frac{2}{\pi} \sigma^2 \frac{k_B T}{\Delta E} \cdot \frac{1}{\omega} \left[\operatorname{atan}(\omega \tau_2) - \operatorname{atan}(\omega \tau_1) \right]$$

$$\sim \underline{\frac{1}{\omega}} \quad \text{for} \quad \tau_2^{-1} \ll \omega \ll \tau_1^{-1}$$

The range of frequencies is, for example,

$$\frac{\tau_2}{\tau_1} = e^{\Delta E / k_B T} \approx 5 \cdot 10^6 \quad \text{for} \quad \Delta E = 0.4\,\text{eV}$$

$$T = 300\,\text{K}.$$