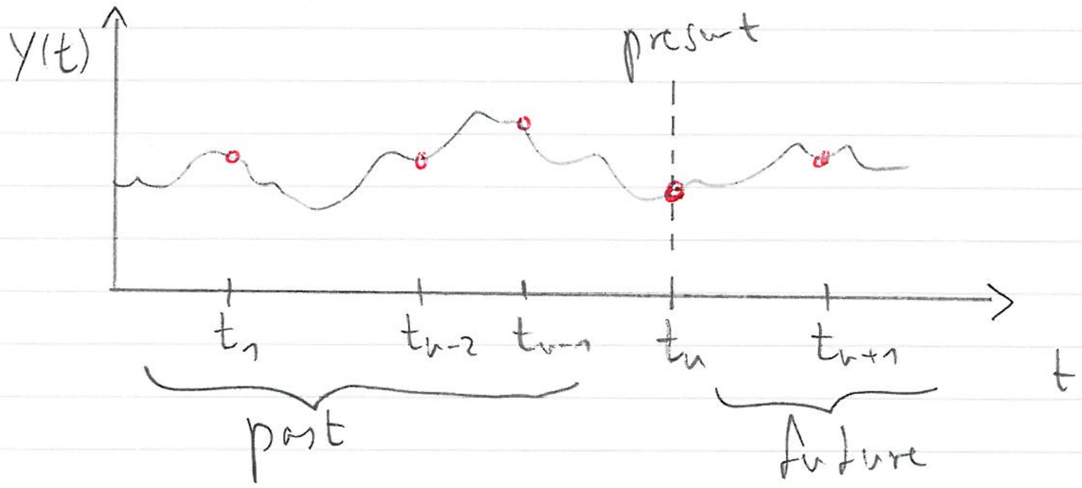
3° Markov processes

a) The Markov property

Def.: A stochastic process is Markovian if

M $$P_{1|n}(y_{n+1}, t_{n+1} | y_1, t_1; \dots; y_n, t_n) = P_{1|1}(y_{n+1}, t_{n+1} | y_n, t_n)$$

for all n, $\{y_i\}$, $\{t_i\}$. In words: conditioned on the present (y_n, t_n), the past and the future are independent.



Compared to complete independence

$$P_{1|n}(y_{n+1}, t_{n+1} | y_1, t_1; \dots; y_n, t_n) = P_1(y_{n+1}, t_{n+1})$$

which is not possible for "reasonable" (e.g., continuous) processes, M implies that Markov processes have a minimal, 1-step memory.

A Markov process is completely determined by

- the 1-point distribution fct. $P_1(y, t)$
- the transition probability $P_{1|1}(y, t \mid y', t')$

Using M, all higher order distribution fct. can be constructed:

$$P_2(y_1, t_1; y_2, t_2) = P_{1|1}(y_2, t_2 \mid y_1, t_1)\, P_1(y_1, t_1)$$

$$P_3(y_1, t_1; y_2, t_2; y_3, t_3) = P_{1|2}(y_3, t_3 \mid y_1, t_1; y_2, t_2) \times$$
$$\times P_2(y_1, t_1; y_2, t_2) \overset{M}{=} P_{1|1}(y_3, t_3 \mid y_2, t_2)\, P_2(y_1, t_1; y_2, t_2) =$$
$$= P_{1|1}(y_3, t_3 \mid y_2, t_2)\, P_{1|1}(y_2, t_2 \mid y_1, t_1)\, P_1(y_1, t_1)$$

so that in general

$$P_n(y_1, t_1; \ldots; y_n, t_n) = \left[\prod_{k=1}^{n-1} P_{1|1}(y_{k+1}, t_{k+1} \mid y_k, t_k)\right] P_1(y_1, t_1)$$

Remarks: (i) First order ODE's are deterministic Markov processes:

$\dot{x} = f(x,t) \Rightarrow$ solution with $x(t_0) = x_0$ is

$$x(t) = \phi(x_0, t_0, t)$$

$\Rightarrow$ transition probability

$$P_{1|1}(x,t \mid x_0, t_0) = \delta(x - \phi(x_0, t_0, t))$$

(ii) The transition probability evolves the 1-step distribution function in time:

Suppose $P_1(y_1, t_1)$ is known, then for $t_2 > t_1$

$$P_1(y_2, t_2) = \int dy_1\, P_2(y_1, t_1; y_2, t_2) =$$

T

$$= \int dy_1\, P_{1|1}(y_2, t_2 \mid y_1, t_1)\, P_1(y_1, t_1)$$

which is not restricted to Markov processes.

b) The Chapman-Kolmogorov equation

Consider three time points $t_1 < t_2 < t_3$, then

$$P_2(y_3,t_3;y_1,t_1) = \int dy_2\, P_3(y_3,t_3;y_2,t_2;y_1,t_1) =$$

$$= P_1(y_1,t_1) \int dy_2\, P_{1|1}(y_3,t_3;y_2,t_2)\, P_{1|1}(y_2,t_2;y_1,t_1)$$

and dividing both sides by $P_1(y_1,t_1)$ we have

CK

$$P_{1|1}(y_3,t_3|y_1,t_1) = \int dy_2\, P_{1|1}(y_3,t_3|y_2,t_2) \times$$
$$\times P_{1|1}(y_2,t_2|y_1,t_1)$$

which is a consistency condition on the time evolution induced by $P_{1|1}(y,t|y',t')$:

$$\begin{tikzcd} P_1(y_1,t_1) \arrow[r, "P_{1|1}"] \arrow[rr, bend right, "P_{1|1}"'] & P_1(y_2,t_2) \arrow[r, "P_{1|1}"] & P_1(y_3,t_3) \end{tikzcd}$$

Any pair of functions P_1, $P_{1|1}$ that satisfy T and CK uniquely define a Markov process.

Examples: (i) Poisson process: The state space are the nonnegative integers, $Y(t) = N_{[at]} = 0,1,2\ldots$ and the transition probability is

$$P_{1|1}(N_2,t_2 \mid N_1,t_1) = \text{Prob}[N_2 - N_1 \text{ events in } (t_1,t_2)]$$

$$= e^{-\rho(t_2-t_1)} \frac{[\rho(t_2-t_1)]^{N_2-N_1}}{(N_2-N_1)!}$$

and the initial condition $P_1(N,0) = \delta_{N,0}$.

(ii) Wiener process: $[Y(t) \in \mathbb{R},\ t \geq 0,\ Y(0)=0]$

Transition probability:

$$\begin{cases} P_{1|1}(y_2,t_2 \mid y_1,t_1) = \dfrac{1}{\sqrt{2\pi(t_2-t_1)}} \exp\left(-\dfrac{(y_2-y_1)^2}{2(t_2-t_1)}\right) \\ P_1(y,0) = \delta(y) \end{cases}$$

Problem: Verify that these processes satisfy CK-equation

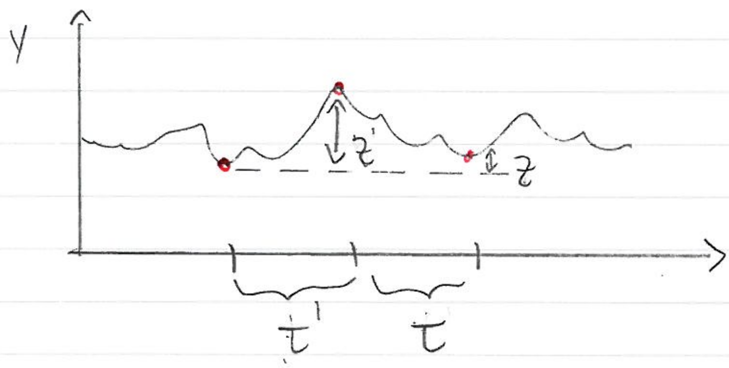
c) Markov processes with stationary increments

A Markov process is called homogeneous if $P_{1|1}(y,t\,|\,y',t')$ depends only on the time difference $t-t'$. The examples (i), (ii) above are in addition translationally invariant in y, i.e.

$$P_{1|1}(y,t\,|\,y',t') = \Phi(y-y', t-t')$$

The CK-eq. then becomes a functional equation for Φ:

$$\Phi(z, \tau+\tau') = \int dz' \, \Phi(z',\tau')\, \Phi(z-z',\tau)$$



Introducing the generating fct.

$$G(k,\tau) = \int dz \, \Phi(z,\tau)\, e^{ikz}$$

the convolution becomes

$$G(k, \tau + \tau') = G(k,\tau)\, G(k,\tau')$$

analogous to the addition of i.i.d. RV's.
The general solution is $G(k,\tau) = e^{-\tau\,\phi(k)}$.
A particular class of processes are the
Lévy processes

$$\Phi_\alpha(z,\tau) = \frac{1}{2\pi}\int dk\, e^{-ikz}\, e^{-c\tau|k|^\alpha}$$

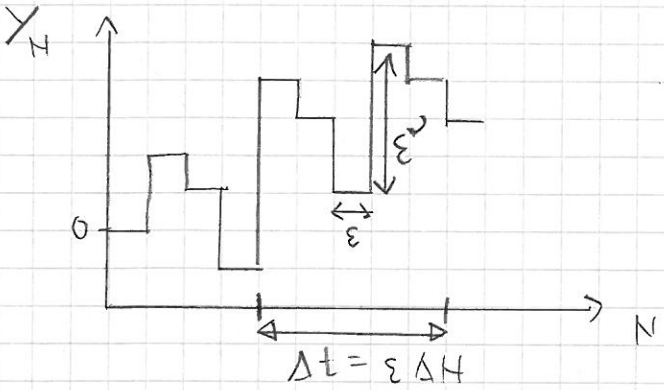
$$G_\alpha(k,\tau) = e^{-c\tau|k|^\alpha}$$

which arise as a continuum limit of
the Lévy flights defined in I. 2°.

Special cases: $\alpha = 2 \Rightarrow$ Wiener process

$\alpha = 1$: $F_1(z,\tau) = \frac{c\tau}{\pi}\frac{1}{c^2\tau^2 + z^2}$ Cauchy process

The continuum limit requires a rescaling
of time and "space" Y:



$$\begin{cases} Y_N = \sum_{i=1}^{N} X_i \\ f(x) \sim |x|^{-(\alpha+1)}, \text{ symmetric} \end{cases}$$

<u>Continuum limit</u> : $\varepsilon \to 0$, $N \to \infty$ at fixed $t = N\varepsilon$

To obtain a non-degenerate limit, define the stochastic process

$$Y(t) := \lim_{\varepsilon \to 0} \varepsilon^\nu Y_{t/\varepsilon} = t^\nu \lim_{N \to \infty} N^{-\nu} Y_N$$

We know that $\nu = \max[1/2, 1/\alpha]$. For $\alpha < 2$

$$\lim_{N\to\infty} N^{-1/\alpha} Y_N = z \quad \text{Lévy RV with pdf}$$

$$f_\alpha(z) = \frac{1}{2\pi} \int dk\, e^{-ikz} e^{-C|k|^\alpha}$$

$$= L_{\alpha,0}(z) \quad \text{Lévy stable law}$$

Since $\Phi_\alpha(y,t)$ is the pdf of the rescaled increments of the process, it follows that

$$\Phi_\alpha(y,t) = t^{-1/\alpha} L_{\alpha,0}(y/t^{1/\alpha}) =$$

$$= t^{-1/\alpha} \cdot \frac{1}{2\pi} \int dk\, e^{-iky/t^{1/\alpha}} e^{-C|k|^\alpha} =$$

$$= \frac{1}{2\pi} \int dk\, e^{-iky} e^{-Ct|k|^\alpha}. \quad \square$$

d) Stationary Markov processes

In the stationary case we have

$$\left.\begin{aligned} P_1(y,t) &= P_1(y), \\ P_{1|1}(y,t\,|\,y',t') &=: T_{t-t'}(y\,|\,y') \end{aligned}\right\}$$

and the relations T and CK reduce to

CK $$T_{\tau+\tau'}(y\,|\,y') = \int dy''\, T_\tau(y\,|\,y'')\, T_{\tau'}(y''\,|\,y')$$

T $$\underline{P_1(y) = \int dy'\, T_\tau(y\,|\,y')\, P_1(y')}$$

$\Rightarrow$ P_1 is the right eigenfunction of the time evolution operator with eigenvalue 1.

Moreover

$$\int dy\, T_\tau(y\,|\,y') = \int dy\, \frac{P_2(y,t;\,y',t+\tau)}{P_1(y',t+\tau)} =$$

$$= P_1(y')/P_1(y') = 1$$

$\Rightarrow$ the left eigenfunction of $T_\tau(y\,|\,y')$ is the constant function.

Example: Ornstein-Uhlenbeck process

$$\left.\begin{aligned} p_1(y) &= \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2} \\ T_\tau(y|y') &= \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}} \exp\left[-\frac{(y-e^{-\tau}y')^2}{2(1-e^{-2\tau})}\right] \end{aligned}\right\}$$

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e) Markov chains

A Markov chain is a homogeneous Markov process with
- discrete state space (finite or infinite)
- discrete time $t \in \mathbb{Z}$.

The CK-equation then implies

$$T_{\tau+1}(y|y') = \underbrace{\sum_{y''} T_\tau(y|y'')\, T_1(y''|y')}_{\text{matrix multiplication}}$$

$$\Rightarrow \quad T_\tau(y|y') = (T_1)^\tau (y|y')$$

where the 1-step transition matrix T_1 satisfies

- $T_1(y|y') \geq 0$
- $\sum_y T_1(y|y') = 1$

In the finite dimensional case such matrices are called stochastic.

The stationary distribution $P_s(y)$ is the right eigenvector of T_1

$$\sum_{y'} T_1(y|y')\, P_s(y') = P_s(y)$$

and is guaranteed to exist in the finite-dim. case. Moreover the Perron-Frobenius theorem implies uniqueness of P_s and convergence,

$$\lim_{t\to\infty} (T_1)^t P_1(y,0) = P_s(y)$$

for any $P_1(y,0)$, provided T_1 is irreducible (= every state is connected to every other state).

f) Stationary, Gaussian Markov processes

Recall the following facts:

- A Gaussian process is uniquely specified by $K(t_1, t_2)$ and $\langle Y(t) \rangle$
- For a stationary process $\langle Y(t) \rangle = \bar{Y} = \text{const.}$ and $K(t_1, t_2) = K(|t_1 - t_2|)$
- By shifting and rescaling $Y(t)$ we can achieve that $\bar{Y} = 0, \quad \sigma^2 = K(0) = 1$

$\Rightarrow$ a stationary Gaussian process is uniquely determined by a single, symmetric function $K(\tau)$ with $K(0) = 1$.

Under what condition on $K(\tau)$ is a stationary, Gaussian process also Markovian?

Theorem (Doob): The Ornstein-Uhlenbeck process is the unique stationary, Gaussian Markov process.

Sketch of proof: For a stationary, Gaussian process with $\bar{Y} = 0$, $\sigma^2 = 1$ we have

$$P_1(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

and the most general ansatz for the transition probability reads

$$T_\tau(y|y') = D \exp\left[-\frac{1}{2}\left(Ay^2 + 2Byy' + Cy'^2\right)\right]$$

with τ-dependent coefficients A, B, C, D.

Using the condition

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} dy\, T_\tau(y|y') = 1 \\ &\int_{-\infty}^{\infty} dy'\, T_\tau(y|y')\, P_1(y') = P_1(y) \end{aligned}\right\}$$

one finds that

$$T_\tau(y|y') = \frac{1}{\sqrt{2\pi(1-\kappa^2)}} \exp\left(-\frac{(y-\kappa y')^2}{2(1-\kappa^2)}\right)$$

with a single parameter $\kappa = \kappa(\tau)$. Inserting this into the CK-equation yields

$$\kappa(\tau+\tau') = \kappa(\tau)\,\kappa(\tau') \quad \forall \; \tau, \tau'$$

$$\Rightarrow \quad \kappa(\tau) = e^{-\lambda|\tau|} \quad , \; \lambda > 0 .$$

For $\lambda = 1$ this is the Ornstein-Uhlenbeck process. □

4° Continuity, differentiability and zero crossings

a) Notions of continuity for stochastic processes

A. Every realization $Y_x(t)$ is everywhere continuous

B. Almost sure continuity (cont. in probability)

"Prob [$Y_x(t)$ is discontinuous] = 0"

For Markov processes this can be quantified by the Lindeberg condition

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_{|y_1-y_2|>\varepsilon} dy_2 \; P_{1|1}(y_2, t+\tau \,|\, y_1, t) = 0 \quad \forall \; \varepsilon > 0$$

$\Rightarrow$ probability of a jump in an interval of size τ vanishes faster than τ.

C. <u>Continuity in the mean</u>:

$$\lim_{\varepsilon \to 0} \langle (Y(t+\varepsilon) - Y(t))^2 \rangle = 0$$

For a stationary process we have

$$\langle (Y(t+\varepsilon) - Y(t))^2 \rangle = 2 \langle Y^2 \rangle - 2 \langle Y(t+\varepsilon) Y(t) \rangle$$

$$= 2 (K(0) - K(\varepsilon))$$

$\Rightarrow$ continuity in the mean implies that $K(\tau)$ is continuous at $\tau = 0$.

The three notions of continuity are clearly different, with A. $\Rightarrow$ B. $\Rightarrow$ C.

<u>Examples:</u>

(i) <u>Telegraph process:</u> $K(\tau) = e^{-2\rho(\tau)}$

$\Rightarrow$ process is continuous in the mean, because the discontinuities form a set of zero measure.

On the other hand

$$\text{Prob}[\,|Y(t+\tau) - Y(t)| > \varepsilon\,] =$$

$= \text{Prob}[$ odd number of jumps in $(t, t+\tau)\,] =$

$$= \frac{1}{2}(1 - e^{-2\rho\tau})$$

Problems

$$\Rightarrow \lim_{\tau\to 0} \frac{1}{\tau} \mathrm{Prob}[\,|Y(t+\tau) - Y(t)| > \varepsilon\,] = \rho > 0$$

$\Rightarrow$ process is discontinuous in the sense of B.

(ii) <u>Ornstein-Uhlenbeck process:</u> $K(\tau) = K(0)\, e^{-\lambda|\tau|}$

$\Rightarrow$ continuity in the mean holds as in (i).

On the other hand, the transition probability for $\tau \to 0$ is

$$T_\tau(y\,|\,y') = \frac{1}{\sqrt{4\pi\tau}}\, e^{-\frac{(y-y')^2}{4\tau}}$$

$$\Rightarrow \mathrm{Prob}[\,|Y(t+\tau) - Y(t)| > \varepsilon\,] =$$

$$= \frac{1}{\sqrt{\pi\tau}} \int_\varepsilon^\infty dy\, e^{-y^2/4\tau} = \left[x = \frac{y}{2\sqrt{\tau}}\right]$$

$$= \frac{2}{\sqrt{\pi}} \int_{\varepsilon/2\sqrt{\tau}}^\infty dx\, e^{-x^2} \underset{\tau\to 0}{\simeq} \frac{2}{\sqrt{\pi}} \frac{\sqrt{\tau}}{\varepsilon}\, e^{-\varepsilon^2/4\tau}$$

$$\Rightarrow \lim_{\tau\to 0} \frac{1}{\tau} \mathrm{Prob}[\,|Y(t+\tau) - Y(t)| > \varepsilon\,] =$$

$$= \lim_{\tau\to 0} \frac{2}{\sqrt{\pi}} \frac{1}{\varepsilon\sqrt{\tau}}\, e^{-\varepsilon^2/4\tau} = 0$$

$\Rightarrow$ process is continuous in the sense of B.

The same argument holds for the Wiener process (Brownian motion).

(iii) Lévy-processes:

In this case $P_{1|1}(y, t+\tau \mid y', t) = \Phi_\alpha(y-y', \tau)$ with

$$\Phi_\alpha(y, \tau) = \tau^{-1/\alpha} L_{\alpha,0}(y/\tau^{1/\alpha})$$

$\Rightarrow$ $\mathrm{Prob}[\,|Y(t+\tau) - Y(t)| > \varepsilon\,] =$

$$= 2\int_\varepsilon^\infty dy\, \Phi_\alpha(y, \tau) = 2\int_{\varepsilon/\tau^{1/\alpha}}^\infty dx\, L_{\alpha,0}(x)$$

For large arguments the Lévy stable pdf behaves as

$$L_{\alpha,0}(x) \sim |x|^{-(\alpha+1)}$$

$$\Rightarrow \mathrm{Prob}[\,|Y(t+\tau)-Y(t)| > \varepsilon\,] \sim \int_{\varepsilon/\tau^{1/\alpha}}^\infty dx\, x^{-(\alpha+1)}$$

$$\sim \left(\frac{\varepsilon}{\tau^{1/\alpha}}\right)^{-\alpha} = \frac{\tau}{\varepsilon^\alpha}$$

$$\Rightarrow \lim_{\tau\to 0} \frac{1}{\tau}\, \mathrm{Prob}[\,|Y(t+\tau) - Y(t)| > \varepsilon\,] \sim \varepsilon^{-\alpha} > 0$$

$\Rightarrow$ Lévy process is discontinuous in the sense of B., and jumps of size $> \varepsilon$ occur with probability $\sim \varepsilon^{-\alpha}$.

Continuity in the mean cannot be applied, because $\kappa(\tau)$ does not exist.