b) Differentiability of stochastic processes

The derivative of a stochastic process $Y(t)$ is

$$Y' = \frac{dY}{dt} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(Y(t+\varepsilon) - Y(t)\right)$$

provided the limit exist. For differentiability in the mean require

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \left\langle \left(\frac{Y(t+\varepsilon_1) - Y(t)}{\varepsilon_1} - \frac{Y(t+\varepsilon_2) - Y(t)}{\varepsilon_2} \right)^2 \right\rangle = 0$$

for any sequence $\varepsilon_1, \varepsilon_2$.

- stationary processes:

$$\left.\begin{aligned} &\langle (Y(t+\varepsilon_1) - Y(t))^2 \rangle = 2\left(K(0) - K(\varepsilon_1)\right) \\ &\langle (Y(t+\varepsilon_1) - Y(t))(Y(t+\varepsilon_2) - Y(t)) \rangle = \\ &= K(\varepsilon_1 - \varepsilon_2) - K(\varepsilon_1) - K(\varepsilon_2) + K(0) \end{aligned}\right\}$$

Assume that $K(\tau)$ is twice differentiable at $\tau = 0$. Then it follows that $K'(0) = 0$ and

$$\langle (Y(t+\varepsilon_1) - Y(t))^2 \rangle \longrightarrow -\varepsilon_1^2 K''(0)$$

$$\langle (Y(t+\varepsilon_1) - Y(t))(Y(t+\varepsilon_2) - Y(t)) \rangle \longrightarrow$$

$$\left(\frac{1}{2}(\varepsilon_1 - \varepsilon_2)^2 - \frac{1}{2}\varepsilon_1^2 - \frac{1}{2}\varepsilon_2^2 \right) K''(0) = -\varepsilon_1 \varepsilon_2 K''(0)$$

$$\Rightarrow \lim_{\varepsilon_1, \varepsilon_2 \to 0} \left\langle \left(\frac{1}{\varepsilon_1}(Y(t+\varepsilon_1) - Y(t)) - \frac{1}{\varepsilon_2}(Y(t+\varepsilon_2) - Y(t)) \right)^2 \right\rangle =$$

$$-K''(0) + 2K''(0) - K''(0) = \underline{0}$$

Similarly it can be shown that existence of $K''(0)$ is a necessary condition for the process to be differentiable.

- general processes: Differentiability requires the existence of $\frac{\partial^2}{\partial t_1 \partial t_2} K(t_1, t_2)$.

Remarks:

(i) Markov processes are not differentiable

Example 1: Ornstein-Uhlenbeck-process

$$K(\tau) = e^{-|\tau|} \Rightarrow K''(0) = -\infty$$

Example 2: Wiener process

We know that $\langle (Y(t) - Y(t'))^2 \rangle = |t - t'|$, $Y(0) = 0$

$$\Rightarrow K(t, t') = \langle Y(t) Y(t') \rangle =$$

$$= \frac{1}{2} \left[\langle Y(t)^2 \rangle + \langle Y(t')^2 \rangle - \langle (Y(t) - Y(t'))^2 \rangle \right]$$

$$= \frac{1}{2} \left(t + t' - |t - t'| \right) = \begin{cases} t' & t > t' \\ t & t < t' \end{cases}$$

$$= \min(t, t')$$

$$\Rightarrow \frac{\partial}{\partial t} K(t, t') = \left. \begin{cases} 1 & t < t' \\ 0 & t > t' \end{cases} \right\} = \Theta(t' - t)$$

$$\Rightarrow \frac{\partial^2}{\partial t \, \partial t'} K(t, t') = \delta(t' - t) \neq 0 .$$

(ii) <u>Spectral characterization</u>

We know from the Wiener-Khinchin-theorem that

$$K(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\tau}\, S(\omega)$$

for stationary processes

$$\Rightarrow \quad K''(0) = -\frac{1}{2}\int_{-\infty}^{\infty} d\omega\, \omega^2\, S(\omega)$$

$\Rightarrow$ differentiable if 2nd moment of the frequency spectrum exists.

For $S(\omega) \sim \omega^{-\beta}$ this requires $\beta > 3$.

For $1 < \beta < 3$ we know that

$$K(0) - K(\tau) \sim |\tau|^{\beta-1}, \quad \tau \to 0$$

$$\Rightarrow \quad \langle (Y(t+\tau) - Y(t))^2 \rangle \sim |\tau|^{\beta-1}$$

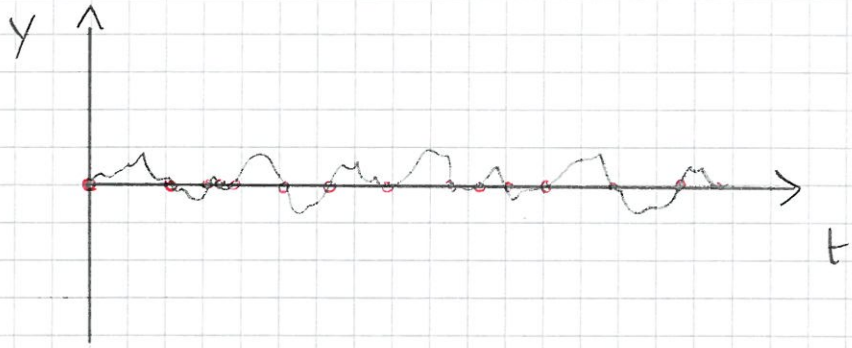
$$\Rightarrow \quad \lim_{\varepsilon \to 0} \frac{\langle (Y(t+\varepsilon) - Y(t))^2 \rangle}{\varepsilon^{2h}} = \begin{cases} 0 & h < \frac{1}{2}(\beta-1) \\ \infty & h > \frac{1}{2}(\beta-1) \\ O(1) & h = \frac{1}{2}(\beta-1) \end{cases}$$

Such a function is called <u>Hölder continuous with index $H = \frac{1}{2}(\beta-1)$</u>. Differentiability requires $H \geq 1$. Markov processes have $H = \frac{1}{2}$.

c) Statistics of zero crossings

Consider stationary, Gaussian process with $\langle Y \rangle = 0$.

What can be said about the set $\{t_i\}$ of zero crossings with $Y(t_i) = 0$?



defines a point process

In particular, under what conditions is there a finite density ρ_0 of zero crossings, in the sense of

$$P_1(\tau) = \text{Prob}[\,1 \text{ zero crossing in } (0,\tau)] \simeq \rho_0 |\tau|$$

for $\tau \to 0$?

We compute the related quantity

$$P_{odd}(\tau) = \text{Prob}[\text{ odd number of zeroes in } (0,\tau)] =$$

$$= P_1(\tau) + P_3(\tau) + P_5(\tau) \simeq P_1(\tau), \quad \tau \to 0.$$

Clearly

$$P_{odd}(\tau) = \text{Prob}[\,Y(t)\,Y(t+\tau) < 0\,]$$

and $Y(t) =: Y_1$, $Y(t+\tau) =: Y_2$ are Gaussian RV's with

$$\left.\begin{aligned} \langle Y_1 \rangle &= \langle Y_2 \rangle = 0 \\ \langle Y_1^2 \rangle &= \langle Y_2^2 \rangle = K(0) = 1 \\ \langle Y_1 Y_2 \rangle &= K(\tau) =: K \end{aligned}\right\}$$

$\Rightarrow$ joint probability density is

$$f(y_1, y_2) = \frac{1}{2\pi\sqrt{1-k^2}} \exp\left[-\frac{1}{2}\,\frac{y_1^2 + y_2^2 - 2k y_1 y_2}{1-k^2}\right]$$

Using this we compute

$$P_{odd}(\tau) = \mathrm{Prob}[\,Y_1 Y_2 < 0\,] = \mathrm{Prob}[\,Y_1/Y_2 < 0\,] =$$

$$= \int_{-\infty}^{0} dz\; f_Z(z)$$

where f_Z is the pdf of $Z = Y_1/Y_2$. One finds

$$f_Z(z) = \frac{1}{\pi}\,\frac{\sqrt{1-k^2}}{(z-k)^2 + 1 - k^2}$$

Cauchy-distribution centered around $z = k$

$$\Rightarrow \quad P_{odd}(\tau) = \frac{1}{2} - \frac{1}{\pi}\,\mathrm{atan}\left(\frac{k(\tau)}{\sqrt{1-k(\tau)^2}}\right)$$

For $\tau \to 0$ we have $k(\tau) \simeq 1 - c\,|\tau|^{\beta-1}$

$$\Rightarrow \frac{k(\tau)}{\sqrt{1-k^2}} \simeq \frac{1 - c\,|\tau|^{\beta-1}}{\sqrt{2c\,|\tau|^{\beta-1}}} \simeq \frac{1}{\sqrt{2c}\;|\tau|^{\frac{1}{2}(\beta-1)}}$$

$$= \frac{1}{\sqrt{2c}\,|\tau|^{H}} \to \infty$$

Use the expansion

$$\mathrm{atan}(1/x) \simeq \frac{\pi}{2} - x, \quad x \to 0$$

$$\Rightarrow P_{odd}(\tau) \simeq \frac{1}{\pi}\sqrt{2c}\,|\tau|^{H}, \quad \tau \to 0$$

Two cases:

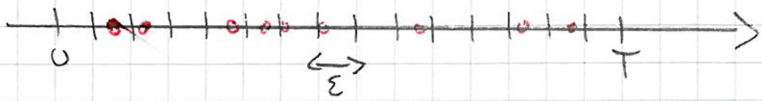
(i) H = 1, process is differentiable: Then $C = -\frac{1}{2} k''(0)$

and $P_{odd}(\tau \to 0) = P_1(\tau \to 0) = \rho_0 |\tau|$

with $\rho_0 = \frac{\sqrt{-k''(0)}}{\pi}$ density of zero crossings (Rice, 1944/45)

(ii) H < 1: In this case the density of zero crossings is not finite; rather, the zeroes form a fractal set of dimension $D = 1 - H$.

To define this notion, consider a set of points on an interval $(0, T)$:



Subdivide $(0, T)$ into intervals of size ε, and denote by $N(\varepsilon)$ the number of intervals required to cover the set. Then the (Hausdorff, capacity) dimension D of the set is defined by

$$N(\varepsilon) \sim \varepsilon^{-D}, \quad \varepsilon \to 0$$

Examples:

- Finite set of N_0 points:

$$\lim_{\varepsilon \to 0} N(\varepsilon) = N_0 \quad \Rightarrow \quad D = 0$$

- continuous covering:

$$N(\varepsilon) = T/\varepsilon \quad \Rightarrow \quad \underline{D = 1}$$

In the present case the probability to find at least one zero in an interval of length ε is proportional to ε^H

$$\Rightarrow \quad N(\varepsilon) \sim \left(\frac{T}{\varepsilon}\right) \varepsilon^H \sim \varepsilon^{-(1-H)} . \qquad \square$$

5° Fractional Brownian motions

Recall: The Wiener process is a non-stationary, Gaussian process with stationary increments.

This implies that the increment process

$$Z_\tau(t) := Y(t+\tau) - Y(t)$$

is a stationary Gaussian process with

$$\langle Z_\tau(t) \rangle = 0 \quad , \quad \langle Z_\tau(t)^2 \rangle = \tau \; ,$$

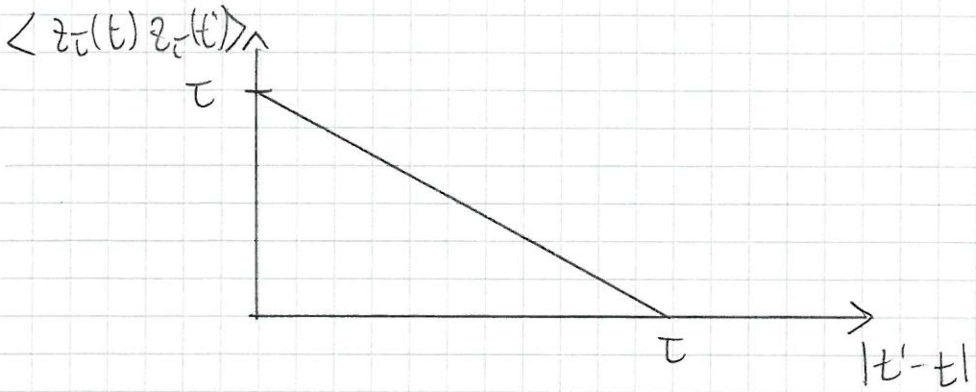
$$\langle Z_\tau(t) Z_\tau(t') \rangle = \langle (Y(t+\tau) - Y(t))(Y(t'+\tau) - Y(t')) \rangle$$

$$= \min(t+\tau, t'+\tau) - \min(t+\tau, t') - \min(t, t'+\tau) +$$

$$+ \min(t, t') = \qquad\qquad (t' > t)$$

$$= t+\tau - \min(t+\tau, t') - t + t =$$

$$= \left\{ \begin{array}{ll} 0 & t+\tau < t' \\ \tau - (t'-t) & t+\tau > t' \end{array} \right\} = \max(\tau - |t-t'|, 0)$$



Mandelbrot & van Ness (1968) introduce fractional Brownian motion (fBM) $B^H(t)$ as the unique Gaussian process with stationary increments

$$z^H_\tau(t) = B^H(t+\tau) - B^H(t)$$

satisfying

$$\left.\begin{array}{ll} \text{(i)} & B^H(0) = 0 \\ \text{(ii)} & \langle z^H_\tau(t)\rangle = 0 \\ \text{(iii)} & \langle (z^H_\tau(t))^2\rangle = \langle (B^H(t+\tau) - B^H(t))^2\rangle = \tau^{2H} \end{array}\right\}$$

- $H \in (0,1)$ is the Hurst-or Hölder exponent and determines the Hölder continuity class of fBM.
- The Wiener process (standard Brownian motion) is $B^{1/2}$ in this notation.

Autocorrelation function of B^H:

$$\langle (B^H(t) - B^H(t'))^2\rangle = \underbrace{\langle B^H(t)^2\rangle}_{|t|^{2H}} + \underbrace{\langle B^H(t')^2\rangle}_{|t'|^{2H}}$$
$$- 2\langle B^H(t)\, B^H(t')\rangle$$
$$= |t-t'|^{2H}$$

$$\Rightarrow \langle B^H(t)\, B^H(t') \rangle = \frac{1}{2}\left\{ |t|^{2H} + |t'|^{2H} - |t-t'|^{2H} \right\}$$

Increment correlation function:

$$\langle z_\tau^H(t)\, z_\tau^H(t') \rangle = \frac{1}{2}\left(|\tau + t - t'|^{2H} + |\tau + t' - t|^{2H} - 2|t-t'|^{2H} \right)$$

$$= |\tau|^{2H}\, \phi(|t-t'|/|\tau|)$$

with $\phi(x) = \frac{1}{2}\left(|1-x|^{2H} + (1+x)^{2H} - 2x^{2H} \right)$

Properties of $\phi(x)$:

- $\phi(x \to 0) \simeq 1 - x^{2H}$
- ϕ has a singularity at $x = 1$, and
  $$\phi(1) = 2^{2H-1} - 1 \begin{cases} > 0 & H > 1/2 \\ < 0 & H < 1/2 \end{cases}$$
- For large x: $\phi(x) \simeq H(2H-1)\, x^{-(2-2H)}$

$\Rightarrow$ fBM is non-Markovian with long-ranged correlations that are persistent ($\phi > 0$) for $H > \frac{1}{2}$ and anti-persistent ($\phi < 0$) for $H < \frac{1}{2}$.

+ figures, examples