The Quadrupolar Pomeranchuk Instability: Interplay of Multiple Critical Modes with Different Dynamics

Quadrupolare Pomeranchuk Instabilität:

Zusammenspiel zweier kritischer Moden
mit unterschiedlicher Dynamik

Diploma Thesis

by

Mario Zacharias

Institute of Theoretical Physics, Department of Physics
Faculty of Mathematics and Natural Sciences
University of Cologne

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Abstract

This work considers the quadrupolar Pomeranchuk instability of an isotropic Fermi liquid in $d = 2$ dimensions. Starting from free electrons subjected to a quadrupolar interaction one can derive an effective Ginzburg-Landau theory for the nematic order parameter [1].

The crucial point of this theory is the presence of two simultanously critical modes, which have different dynamics. There is a ballistic mode with a dynamical exponent $z = 2$ and a Landau-damped mode with $z = 3$. At zero temperature, the ballistic mode is at its upper critical dimension and interactions result in logarithmic singularities in perturbation theory. In contrast, at finite temperatures the $z = 3$ mode dominates thermodynamics, due to the larger available phase space.

Trying to analyze the zero temperature theory one has to resum the arising divergences. In general this is done by means of renormalization group (RG) techniques. But due to the presence of two different dynamical exponents there is the problem, how to rescale frequencies. In the usual RG-scheme this rescaling is done according to $\omega \rightarrow b^z \omega$. Since the model has two different dynamical exponents, the scheme has to be adjusted in the first place. With this RG-scheme the mass correction at zero temperature can be calculated. The resulting universality class is distinct from the Ising- as well as from the XY-model.

At finite temperatures, there is a complex interplay between the two modes. At criticality, the length scale $\xi_T \sim T^{-1/z}$ separates the quantum from the classical regime where each mode has effective dimensions $d + z$ and d, respectively. The presence of two different exponents z yields an overlap regime where the ballistic mode still has its quantum character and the damped mode is already classical. It can be shown that this overlap regime dominates the system at finite temperature and that the theory cannot be reduced to an effective classical model implying a breakdown of "dimensional reduction". The interplay of critical modes leads to a universal temperature dependence of the correlation length.

Finally, the thermodynamical properties of the system are calculated on a Gaussian level. As expected the higher $z = 3$ mode dominates the behavior of the specific heat and the thermal expansion. In contrast, the compressibility of the system is governed by the mode with $z = 2$.

Contents

Chapter 1

Introduction

1.1 Motivation

In most theoretical models of quantum phase transitions the assumption is made that only one time scale, τ, characterizing the physical system exists. At criticality this time scale is given by $\tau \sim \xi^z$ where ξ is the correlations length, and z is the so-called critical dynamical exponent. This assumption is not always true since, for instance, in a metal close to a magnetic phase transition two different time scales exist. On the one hand, there is the time scale of the quasiparticles, and on the other hand the time scale of the magnetic fluctuations. In such systems two different time scales are coupled. In general, an interplay of different dynamics can be expected to yield interesting phenomena. An example of such a system is the transition of an isotropic Fermi liquid to a nematic electron liquid.

In the following work such a isotropic-to-nematic transition in $d = 2$ spatial dimensions is investigated from the disordered side. At the nematic transition the rotational symmetry of the Fermi surface is spontaneously broken. In the ordered state the Fermi surface has only a residual quadrupolar symmetry. This is an example of the so-called Pomeranchuk instability [2].

Following the pioneering work of Oganesyan et al. [1] the system is described by a Landau-Ginzburg-Wilson action for the two-component order parameter (see Sect. 3.4). This action consists of two modes with different dynamics which become critical at the very same point. Moreover, the two modes have even different critical dynamical exponents. On the one hand, there is a mode with $z = 3$ which is subject to damping due to quasiparticle-quasihole excitations. On the other hand, a mode with $z = 2$ exists which is undamped and hence ballistic. These two dynamical exponents lead to two different time scales $\tau_2 \sim \xi^2$ and $\tau_3 \sim \xi^3$ at criticality.

Oganesyan et. al argued that for the critical theory only the damped mode is important, while the mode with $z = 2$ has a higher characteristic energy and plays no role. Furthermore, $z = 3$ scaling implies that interactions between both modes

are irrelevant. In this thesis it is shown that, on the contrary, the system at zero temperature strongly relies on the presence of the $z = 2$ mode. Since this mode has an effective dimension $d + z = 4$, interactions yield logarithmically diverging corrections (see Chap. 4). In addition, interactions of both modes are shown to be crucial at finite temperatures. The interplay between both modes yields unexpected consequences, which disagree with naive scaling arguments (see Sect. 5.3).

The presence of a ballistic $z = 2$ mode, however, relies on the isotropy of the system. For systems with an underlying lattice the rotational symmetry is broken and it remains only a point group symmetry. Therefore, there is no mode with $z = 2$ but only singular points on the Brillouin zone diagonals which are undamped [3].

1.2 Experimental Evidence for Pomeranchuk Instabilities

The quadrupolar Pomeranchuk instability is not only interesting from a theoretical point of view. In fact, there are plenty of systems which show evidence for a spontaneously broken symmetry towards a nematic state. Some of these are believed to originate from a quadrupolar Pomeranchuk instability.

Transport measurements on clean two-dimensional electron gases in high Landau levels show a broken rotational symmetry of the resistivity tensor [4]. This can be explained by the onset of a stripe order at low temperatures. For strong quantum fluctuations, the translational symmetry is restored, yielding the so-called "quantum Hall nematic" phase [5].

In solids, the underlying lattice symmetry breaks the rotational symmetry. Therefore, the system is not isotropic from the outset but only has a point group symmetry. Nevertheless, there are experiments which show a spontaneous point group symmetry breaking which is suspected to originate from a transition towards a nematic state.

An example for such a system is $Sr_3Ru_2O_7$ close to its metamagnetic transition. This layered ruthenate has a quasi-two-dimensional electronic structure and a fourfold symmetric Fermi surface. Transport measurements near the metamagnetic transition show a new narrow phase close to the quantum critical point. The authors of [6] suggest a spin dependent twofold symmetric Fermi surface distortion as an explanation. Also for the "hidden order" of URu_2Si_2 there is a proposed scenario involving the quadrupolar Pomeranchuk instability [7].

In high-T_c superconducting cuprates an anisotropic resistivity tensor was also found. This nematic state can be explained by a certain model of Fermi surface distortions [8]. Alternatively, a model of spatially fluctuating stripe order was proposed to explain the experimental data [9].

In this thesis the nematic transition from an isotropic state is considered, which differs from lattice models as mentioned before. A controlled experiment to study such a transition has not been achieved, yet. A possibility may be systems of ultra-cold atoms. Using Feshbach resonances in the d-wave channel, the direct measurement of the collective excitations, that are responsible for such a transition, would be possible. Changing the interaction strength then leads to a deformation of the atomic cloud. However, the realization of such an experiment is a challenging task.

1.3 Outline

In this thesis an isotropic fermionic system near a quadrupolar Pomeranchuk instability in $d = 2$ spatial dimensions is studied.

In Chap. 2 the concept of Fermi liquid theory is briefly explained. After introducing the basic ideas, the transport equation and the ground state excitations are discussed. Focussing on collective excitations, the concept of Landau damping is introduced.

Chap. 3 introduces the system of concern of this thesis. Discussing the circumstances under which the Fermi surface becomes unstable, the focus is drawn on the quadrupolar Pomeranchuk instability. Landau damping is specified for this model, and finally, the effective action for the nematic order parameter is derived.

The analysis of the effective action at zero temperature is done in Chap. 4 by means of renormalization group (RG) techniques. First, the general idea of renormalization group is presented. Thereafter, the general scheme is adapted to the model at hand, taking the presence of two different dynamical exponents into account. With this adapted scheme the mass and vertex corrections are calculated to one-loop order and the zero-temperature correlation length is obtained. Strikingly, the mode with $z = 2$ is found to govern the system at zero temperature and a new universality class is found.

At finite temperatures, in Chap. 5, the method of dimensional reduction is applied to the model and shown to break down. Therefore, the model is again analyzed by means of the previously established RG scheme and the finite temperature correlation length is obtained to be $\xi^{-2} = 16/3 \ln(3/2) T$, which is thus independent of the interaction strength. To check the applied RG scheme, the contribution in second order perturbation theory is calculated, that is most divergent.

Finally, thermodynamics is considered in Chap. 6. After a general introduction into quantum phase transitions, the specific heat, the thermal expansion and the compressibility are derived from scaling theory. The calculation of these quantities on a Gaussian level is performed and compared to the scaling results. While the specific heat and the thermal expansion are governed by the mode with $z = 3$, the compressibility is determined by the $z = 2$ mode.

Chapter 2

Fermi Liquid Theory

In the beginning of the 20th century the characteristic properties of metals were explained, treating the conduction electrons as a non-interacting Fermi gas. At zero temperature the fermions, according to Pauli exclusion principle, are frozen into the lowest energy levels and build the Fermi sphere. At finite temperatures they are distributed according to the well known *Fermi function*

$$f(\epsilon) = \frac{1}{1 + \exp[(\epsilon - \mu)/k_B T]} \tag{2.1}$$

The sharp edge of the Fermi sphere gets smeared out over a region of order $k_B T$, where k_B is the Boltzmann factor. Although neglecting any interactions, this single particle picture explains a variety of physical observables, such as the specific heat.

However, there are many-body effects, which rely crucially on the interaction of the particles. In order to describe liquid ^{3}He Landau in a series of articles ([10], [11] and [12]) proposed a semi-phenomenological approach, known as *Fermi liquid theory*. This powerful model takes interactions seriously into account and is the basis of modern understanding of fermionic systems. Furthermore, it is also capable to explain the success of the single-particle picture.

In the following chapter, the basic ideas and some consequences of Fermi liquid theory are described. In the first section, the main concept is briefly summarized. Focussing on non-equilibrium processes the transport equation is stated and the elementary excitations of the Fermi liquid, namely quasiparticle excitations and collective modes, are explained. As an example Landau's concept of the zero sound mode is presented. Finally, the last section concerns with the mechanism of Landau damping.

Much more far-reaching explanations of the theory of Fermi liquids can be found in various textbooks e.g. [13] and [14].

2.1 Basic Ideas of Fermi Liquid Theory

In the following a system of spinless fermions is considered. To get information about the interacting system, one may think of switching on interactions "adiabatically," , i.e. infinitely slowly. As a definition of a *normal* Fermi liquid one assumes that the ground state of the interacting system evolves continuously from the ground state of the non-interacting system. One has to note that this needs not to be true. For example, the formation of Cooper pairs in a superconductor cannot evolve continuously from the plain wave superposition of a non-interacting Fermi system.

When adding a particle with momentum **p** to the ground state and switching on interactions, this particle will start to perturb the nearby particles; the particle is "dressed" with a self-energy cloud. The idea of Fermi liquid theory is to consider the dressed particle as the new fundamental entity, called *quasiparticle*. The same applies for holes leading to *quasiholes*. Due to the adiabaticity the quantum numbers cannot change throughout the evolution. This one-to-one correspondence between the original particles and the quasiparticles explains the success of the single particle approach of a Fermi gas.

Of course, the quasiparticles have a finite lifetime which is inversely proportional to the squared distance from the Fermi surface. The adiabatic procedure is therefore meaningless for quasiparticles with high energies, since they decay much faster than the interaction is switched on. Hence, the concept is restricted to the vicinity of the Fermi surface, where $\frac{1}{\tau} \sim (\epsilon - \epsilon_F)^2$ is small. This also implies that one cannot obtain the ground state energy, which would involve a summation over all states within the Fermi sea. Nevertheless, one can describe excitations from the (unknown) ground state and formulate a theory for, e.g. , transport properties.

The quasiparticles and quasiholes have a certain distribution function $n_\mathbf{p}$, and the excitations of the system are given by the deviation from the ground state

$$\delta n_\mathbf{p} = n_\mathbf{p} - n_\mathbf{p}^0 \tag{2.2}$$

Close to the ground state one can obtain the energy difference to the ground state energy F_0, of the system formally by carrying out a Taylor expansion up to second order

$$\Delta F = F(n_\mathbf{p}) - F_0 = \sum_\mathbf{p} (\epsilon_\mathbf{p} - \mu)\delta n_\mathbf{p} + \frac{1}{2}\sum_{\mathbf{p},\mathbf{p}'} f_{\mathbf{p},\mathbf{p}'}\, \delta n_\mathbf{p}\delta n_{\mathbf{p}'} + \mathcal{O}(\delta n^3) \tag{2.3}$$

This relation is the very starting point and the center-piece of Fermi liquid theory.

The first functional derivative of F has to vanish at the Fermi surface

$$\left.\frac{\delta F}{\delta n_\mathbf{p}}\right|_{|\mathbf{p}|=p_F} = \epsilon_F - \mu = 0 \tag{2.4}$$

6

due to the stability of the state. Furthermore, one defines the *group velocity*

$$\mathbf{v}_\mathbf{p} = \nabla_\mathbf{p}\epsilon_\mathbf{p} \tag{2.5}$$

The nomenclature will become clear in the following. For isotropic systems one may also define the *effective mass* as

$$m^\star = p_F/v_F \tag{2.6}$$

where p_F and v_F are momentum and velocity at the Fermi surface.

The most important feature of Eq. (2.3) is the presence of the quadratic term and the function $f_{\mathbf{p},\mathbf{p}'}$ which captures the interaction of the quasiparticles. Although it is not possible to compute this function directly, one can obtain how physical properties are modified by its pure existence. However, especially for isotropic systems there are some things one can say about the function $f_{\mathbf{p},\mathbf{p}'}$. First of all, since it is the interaction energy of excited quasiparticles, it always has to be symmetric in the momentum indices. For isotropic systems, one can decompose $f_{\mathbf{p},\mathbf{p}'}$ on the Fermi surface into Legendre polynomials leading to

$$f_{\mathbf{p},\mathbf{p}'} = \sum_l f_l P_l(\cos\theta) \tag{2.7}$$

with θ being the angle between the two momenta $\mathbf{p}$ and $\mathbf{p}'$. The constants f_l are called *Landau parameters*. It is often more convenient to use the *reduced Landau parameters* $F_l = \nu_0 f_l$ where ν_0 is the *density of states* at the Fermi surface. For isotropic systems the density of states is given by

$$\nu_0 = \frac{1}{V}\sum_\mathbf{p}\delta(\epsilon_\mathbf{p} - \mu) \approx \int\frac{\mathrm{d}\Omega_d}{(2\pi)^d}\int\mathrm{d}p\,p^{d-1}\delta\big[v_F(p - p_F)\big]$$

$$= K_d\,m^\star p_F^{d-2} \tag{2.8}$$

where the energy was linearized around the Fermi surface, and $K_d = \int\mathrm{d}\Omega_d/(2\pi)^d$ is the averaged solid angle.

From Eq. (2.3) one can obtain the energy of a quasiparticle by adding a particle with momentum $\mathbf{q}$ to a state with a given distribution $\delta n_\mathbf{p}$. This yields

$$\tilde{\epsilon}_\mathbf{q} = \epsilon_\mathbf{q} + \sum_\mathbf{p} f_{\mathbf{p},\mathbf{q}}\delta n_\mathbf{p} \tag{2.9}$$

The energy $\tilde{\epsilon}_\mathbf{q}$ is called the *local energy* since the distribution $\delta n_\mathbf{p}$ and therefore $\tilde{\epsilon}_\mathbf{q}$ itself may depend on the position. One can furthermore introduce the *local equilibrium distribution* $\tilde{n}_\mathbf{q}^0 = n_\mathbf{q}^0(\tilde{\epsilon}_\mathbf{p} - \mu)$ and the deviation from local equilibrium

$$\delta\tilde{n}_\mathbf{q} = n_\mathbf{q} - \tilde{n}_\mathbf{q}^0 \tag{2.10}$$

By using the Eq. (2.9) one obtains the following relation between the deviation from local equilibrium and from the ground state

$$\delta \tilde{n}_{\mathbf{q}} = \delta n_{\mathbf{q}} - \frac{\partial n^0}{\partial \epsilon_{\mathbf{q}}} \sum_{\mathbf{p}} f_{\mathbf{p},\mathbf{q}} \delta n_{\mathbf{p}} \tag{2.11}$$

For isotropic systems, where one can expand the distribution function into spherical harmonics, one then gets

$$\delta \tilde{n}_{lm} = \left(1 + \frac{F_l}{2l+1} \right) \delta n_{lm} \tag{2.12}$$

This correspondence between local and global equilibrium distribution function is especially helpful in calculations.

2.2 Transport Equation

Drawing the attention to non-equilibrium processes, the momentum distribution function becomes position- and time-dependent. However, one has to take into account that the Heisenberg uncertainty principle prevents the simultaneous knowledge of position and momentum. Thus, a non-equilibrium distribution function depending on both position and momentum can not be obtained. However, assuming to be in the macroscopic limit of long wave lengths and low energies, one may treat this as a classical problem with a distribution function $n_{\mathbf{p}}(\mathbf{r}, t)$. The Fourier transform of the distribution function $n_{\mathbf{p}}(\mathbf{q}, \omega)$ depends now on two different momenta $\mathbf{p}$ and $\mathbf{q}$. To meet the macroscopic limit, it has to be $|\mathbf{q}| \ll k_F$ and $\omega \ll \mu$, while $|\mathbf{p}| \approx k_F$ is at the Fermi surface. The distribution function $n_{\mathbf{p}}(\mathbf{q}, \omega)$ may also be seen as a quasiparticle-quasihole excitation with the momenta $\mathbf{p} \pm \mathbf{q}/2$, respectively.

The second assumption is, that the local energy $\tilde{\epsilon}_{\mathbf{p}}(\mathbf{r})$ acts as the classical Hamiltonian of the system of quasiparticles, implying

$$\dot{\mathbf{r}} = \quad \nabla_{\mathbf{p}} \tilde{\epsilon}_{\mathbf{p}}(\mathbf{r}) \tag{2.13}$$

$$\dot{\mathbf{p}} = -\nabla_{\mathbf{r}} \tilde{\epsilon}_{\mathbf{p}}(\mathbf{r}) \tag{2.14}$$

Using these relations, the well known *Boltzmann equation* with an external force $F_{\mathbf{p}}$ reads

$$\frac{\partial n_{\mathbf{p}}(\mathbf{r})}{\partial t} + \nabla_{\mathbf{r}} n_{\mathbf{p}}(\mathbf{r}) \nabla_{\mathbf{p}} \tilde{\epsilon}_{\mathbf{p}}(\mathbf{r}) - \nabla_{\mathbf{p}} n_{\mathbf{p}}(\mathbf{r}) \nabla_{\mathbf{r}} \tilde{\epsilon}_{\mathbf{p}}(\mathbf{r}) + F_{\mathbf{p}} \nabla_{\mathbf{p}} n_{\mathbf{p}}(\mathbf{r}) = I[n_{\mathbf{p}}(\mathbf{r})] \tag{2.15}$$

The term $I[n_{\mathbf{p}}(\mathbf{r})]$ is the so called *collision integral*, which captures all changes in the distribution function due to quasiparticle collisions. In the following, only the collisionless regime is considered where the collision frequency is going to zero and the collision integral can be dropped.

Since the quasiparticle concept is meaningful only in the vicinity of the Fermi sphere, one has to get a transport equation for the excited quasiparticles. First of all, one may separate the local energy and the distribution function according to

$$n_{\mathbf{p}}(\mathbf{r}, t) = n_{\mathbf{p}}^0 + \delta n_{\mathbf{p}}(\mathbf{r}, t) \tag{2.16}$$

$$\tilde{\epsilon}_{\mathbf{p}}(\mathbf{r}) = \epsilon_{\mathbf{p}} + \sum_{\mathbf{p}'} f_{\mathbf{p}, \mathbf{p}'} \delta n_{\mathbf{p}}(\mathbf{r}) \tag{2.17}$$

In many cases, the external force is proportional to the departure from equilibrium $F_{\mathbf{p}} \sim \delta n_{\mathbf{p}}$. Keeping only first order terms in the deviation from equilibrium one can therefore approximate $n_{\mathbf{p}}$ by $n_{\mathbf{p}}^0$ in the force term, and from Eq. (2.15) one obtains

$$\frac{\partial \delta n_{\mathbf{p}}(\mathbf{r})}{\partial t} + \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \delta n_{\mathbf{p}}(\mathbf{r}) - \nabla_{\mathbf{p}} n_{\mathbf{p}}^0 \sum_{\mathbf{p}'} f_{\mathbf{p}, \mathbf{p}'} \nabla_{\mathbf{r}} \delta n_{\mathbf{p}}(\mathbf{r}) = -F_{\mathbf{p}} \cdot \nabla_{\mathbf{p}} n_{\mathbf{p}}^0 \tag{2.18}$$

For a non-interacting, system one would obtain the first two terms only. The last term on the left hand side can be interpreted as an additional force acting on the ground state particles. This force originates from the spatial inhomogeneity of the excited quasiparticles. Using $\nabla_{\mathbf{p}} n_{\mathbf{p}}^0 = \frac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} \nabla_{\mathbf{p}} \epsilon_{\mathbf{p}}$ and comparing with Eq. (2.9) one concludes that

$$\frac{\partial \delta n_{\mathbf{p}}(\mathbf{r})}{\partial t} + \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \delta \tilde{n}_{\mathbf{p}}(\mathbf{r}) = -F_{\mathbf{p}} \cdot \mathbf{v}_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} \tag{2.19}$$

Here, the term of a group velocity $\mathbf{v}_{\mathbf{p}}$ becomes meaningful as it describes the velocity of the quasiparticle distribution. Notably, only the second term relies on the local equilibrium distribution. This term describes the diffusion of quasiparticles which is dictated by the local energy.

2.3 Excitations

If one substitutes a plane wave ansatz for the departure from equilibrium

$$\delta n_{\mathbf{p}}(\mathbf{r}, t) = \delta n_{\mathbf{p}}(\mathbf{q}, t) \, e^{-i(\mathbf{q} \cdot \mathbf{r} - \omega t)} \tag{2.20}$$

into Eq. (2.18), one obtains for a system without external forces

$$(\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}} - \omega) \delta n_{\mathbf{p}}(\mathbf{q}, t) - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} \sum_{\mathbf{p}'} f_{\mathbf{p}, \mathbf{p}'} \delta n_{\mathbf{p}'}(\mathbf{q}, t) = 0 \tag{2.21}$$

There are two kinds of excitations. On the one hand, one can add a *localized quasiparticle* with momentum $\mathbf{p}_0$ to the ground state, corresponding to a *quasiparticle-quasihole excitation* with momenta $\mathbf{p} \pm \mathbf{p}_0 / 2$. Since the excited quasiparticle changes the local energy the surrounding quasiparticles will be driven out of equilibrium, too. On the other hand, one can distort the whole Fermi surface yielding a coherent motion of quasiparticles which are driven back to their equilibrium distribution. Such excitations are called *collective modes*.

2.3.1 Quasiparticles

Adding a localized quasiparticle with momentum $\mathbf{p}_0$ to the system changes the local energy of the surrounding quasiparticles. The bare quasiparticle is in this way surrounded by a polarization cloud. This can be modelled by the following distribution function

$$\delta n_{\mathbf{p}} = \delta_{\mathbf{p},\mathbf{p}_0} + \xi_{\mathbf{p}} \tag{2.22}$$

The Dirac delta function, $\delta_{\mathbf{p},\mathbf{p}_0}$, resembles the localized quasiparticle, while $\xi_{\mathbf{p}}$ describes the polarization cloud. The transport equation (2.21) for the case $\mathbf{p} = \mathbf{p}_0$ then takes the form

$$(\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}_0} - \omega)(1 + \xi_{\mathbf{p}_0}) - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}_0} \frac{\partial n_{\mathbf{p}_0}^0}{\partial \epsilon_{\mathbf{p}_0}} \left(f_{\mathbf{p}_0,\mathbf{p}_0} + \sum_{\mathbf{p}'} f_{\mathbf{p}_0,\mathbf{p}'} \xi_{\mathbf{p}'} \right) = 0 \tag{2.23}$$

Since the polarization cloud $\xi_{\mathbf{p}}$ and the interaction energy $f_{\mathbf{p},\mathbf{p}'}$ are of the order of $1/N$ where N is the number of particles, one can approximate the transport equation to zeroth order as

$$\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}_0} - \omega = 0 \tag{2.24}$$

This justifies the identification of $\mathbf{v}_{\mathbf{p}_0}$ as the group velocity of the quasiparticles. Moreover, the energy ω and the momentum $\mathbf{q}$ of the excitation are not independent from each other.

For all other momenta $\mathbf{p} \neq \mathbf{p}_0$ one obtains from Eq. (2.21) to first order in $1/N$

$$(\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}} - \omega)\xi_{\mathbf{p}} - \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} \sum_{\mathbf{p}'} f_{\mathbf{p},\mathbf{p}'} \xi_{\mathbf{p}'} - f_{\mathbf{p},\mathbf{p}_0} \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} = 0 \tag{2.25}$$

For $|\mathbf{q}| = 0$ the trivial solution is $\xi_{\mathbf{p}} = 0$, corresponding to a bare quasiparticle-quasihole excitation exactly at the Fermi surface. For finite momenta $|\mathbf{q}| \neq 0$ the solution of this equation is non-vanishing and very complicated. However, comparing this result with Eq. (2.19) one concludes that the interaction with the bare quasiparticle at $\mathbf{p}_0$ acts like an additional force on the ground state particles. The propagation of the polarization has the velocity $\mathbf{v}_{\mathbf{p}_0}$ since it has the same energy ω and momentum $\mathbf{q}$ as the bare particle.

2.3.2 Collective Modes

Quantum fluctuations of the quasiparticles may lead to a distortion of the whole quasiparticle distribution. As a result of such a distortion, the interaction forces between the quasiparticles average no longer to zero. The net force acting on the system drives the quasiparticles back to the equilibrium distribution yielding an

oscillatory coherent motion of the quasiparticles. In the presence of excited quasi-particles, they damp out the collective modes very fast. However, in the collisionless regime where only very few excited quasiparticles exist the damping is negligible.

For collective modes the departure from the equilibrium distribution $\delta n_{\mathbf{p}}$ is a smooth function extending over the whole Fermi surface. For an isotropic systems one can write

$$\delta n_{\mathbf{p}} = \delta(\epsilon_p - \mu)u_{\mathbf{p}} \tag{2.26}$$

where $u_{\mathbf{p}}$ is the displacement of the Fermi surface at momentum $\mathbf{p}$. Furthermore, it is

$$\frac{\partial n_{\mathbf{p}}^0}{\partial \epsilon_{\mathbf{p}}} = -\delta(\epsilon_p - \mu) \tag{2.27}$$

Defining θ as the angle between the momentum $\mathbf{q}$ and the group velocity $\mathbf{v}_{\mathbf{p}}$ and introducing the variable $s = \frac{\omega}{v_F q}$ the transport equation (2.21) reads

$$(\cos\theta - s)u_{\mathbf{p}} = -\cos\theta \sum_{\mathbf{p}'} \delta(\epsilon_{p'} - \mu)f_{\mathbf{p},\mathbf{p}'}u_{\mathbf{p}'} = \cos\theta \int \frac{\mathrm{d}\Omega'}{(2\pi)^d}\frac{\nu_0}{K_d}f_{\mathbf{p},\mathbf{p}'}u_{\mathbf{p}'}$$

$$= \frac{\cos\theta}{S_{d-1}} \int \mathrm{d}\Omega'\, F_{\mathbf{p},\mathbf{p}'}u_{\mathbf{p}'} \tag{2.28}$$

where $F_{\mathbf{p},\mathbf{p}'} = \nu_0 f_{\mathbf{p},\mathbf{p}'}$ is the reduced interaction and S_{d-1} is the surface of the unit sphere in d-dimensions.

In three dimensions, one can expand $u_{\mathbf{p}}$ into spherical harmonics $Y_{lm}(\theta, \phi)$. This leads to the conclusion that the different values of m are decoupled while the integration over Ω' yields a mixing of different l values. Therefore, m is a quantum number classifying different collective modes, starting with the *longitudinal* ($m = 0$), the *transverse* ($m = 1$) and the *quadrupolar* ($m = 2$) modes. In two dimensions the expansion into Legendre polynomials P_l shows, that the different l values are coupled as well.

A very simple example of a collective mode is the so called *zero sound*, discovered by Landau. Since the ordinary sound is obtained in a state of local equilibrium, the sound frequency ω has to be much smaller than the collision frequency ν. However, the collision frequency is proportional to the squared distance from the Fermi sphere, which implies $\omega \ll (\epsilon - \epsilon_F)^2 \sim T^2$. Therefore ordinary sound propagation of a given frequency ω cannot exist at arbitrary low temperatures. Nevertheless, there are collective modes propagating through the system.

For a constant interaction, $F_{\mathbf{p},\mathbf{p}'} = F$, Eq. (2.28) has the simple solution

$$u_{\mathbf{p}} = \frac{\cos\theta}{s - \cos\theta} \underbrace{\int \mathrm{d}\Omega' \frac{F}{S_d}u_{\mathbf{p}'}}_{=\text{const}} \tag{2.29}$$

11

Substituting this result back into Eq. (2.28) one obtains after integration an equation which determines the eigenvalues s

$$\frac{1}{F} = \frac{1}{2} \int_{-1}^{1} \mathrm{d}(\cos\theta) \frac{\cos\theta}{\cos\theta - s}$$
$$= \frac{s}{2} \ln\left(\frac{s+1}{s-1}\right) - 1 \tag{2.30}$$

Here, the integration was performed by assuming an infinitesimal imaginary part of the eigenvalue s. There are three possible regimes for the interaction F. For $F > 1$ one gets a real solution $s > 1$. If the interaction energy is attractive $F < 0$, one obtains a complex solution which becomes purely imaginary $s = i|s|$ for strong attraction, $F < -1$. The consequences of these solutions are explained in the following.

2.4 Landau Damping of Collective Modes

So far, collective modes, once they are created by some internal fluctuation, are never-ending oscillations around the equilibrium position. But, of course, there are damping mechanisms of these oscillations.

The first one is the collision of excited quasiparticles. An effective momentum transfer between the colliding particles changes the local energy in the neighborhood and therefore disrupts the self-consistent interaction field which drives the oscillation. However, in the collisionless regime such events take place only rarely and this kind of damping is negligible.

The second damping mechanism is the so-called *Landau damping*. For a collective mode with momentum $\mathbf{q}$ and energy ω one can consider a quasiparticle with momentum $\mathbf{p}$ such that it meets the condition $\mathbf{q} \cdot \mathbf{v_p} = \omega$. This means that the *phase velocity* $|\omega|/|\mathbf{q}|$ of the collective mode is just equal to the velocity of the quasiparticle in direction $\hat{\mathbf{q}}$. Therefore, a steady energy transfer between the mode and the quasiparticle takes place. If the velocity of the quasiparticle is slightly smaller than $\mathbf{p}$, it absorbs energy from the mode, whereas the mode gains energy from particles with slightly larger momenta. Since statistics tell that at low temperatures there are more particles with smaller momenta, in total, energy is transferred from the mode to the quasiparticles. This mechanism leads thereby to an effective damping of the collective mode. Another way to look at this damping mechanism is the excitation of a quasiparticle-quasihole pair.

Of course, such a process is only possible if the phase velocity is smaller than the Fermi velocity, since there are no quasiparticles with higher velocities. Therefore Landau damping is only important for $|s| < 1$.

As an example, one may consider the solution for zero sound derived in Eq. (2.30). For values $0 > F > -1$ one obtains a complex solution for s as stated above, which

corresponds to a complex frequency ω. Substituted in the ansatz (2.20) yields

$$\delta n_{\mathbf{p}}(\mathbf{r}) = \delta n_{\mathbf{p}}(\mathbf{q}) e^{-i(\mathbf{q}\cdot\mathbf{q} - t\,\mathrm{Re}\,\omega)} e^{-t\,\mathrm{Im}\,\omega} \tag{2.31}$$

where the last term damps out the free oscillation. For pure imaginary solutions the collective mode is either overdamped or becomes unstable. This feature is explored further in the next chapter.

Chapter 3

The Quadrupole Pomeranchuk Instability

In this chapter the model of a Pomeranchuk instability, especially the one with quadrupolar symmetry $(l = 2)$, is presented.

In the first part, the mechanism of the Pomeranchuk instability is introduced. Starting from Fermi liquid theory the destabilization of an isotropic Fermi surface due to collective modes is explained and the instability criteria for the Landau parameters F_l are derived. Further on, focussing on the quadrupolar Pomeranchuk instability the Landau damping mechanism for such an anisotropic Fermi surface distortion and its restrictions are illustrated.

In the last part, following Oganesyan [1], a two-component order parameter and the microscopic model are proposed. After performing a Hubbard-Stratonovich transformation and integrating over the microscopic degrees of freedom an effective Landau-Ginzburg action for the order parameter is derived. This effective action has two coupled modes with different dynamics and a quartic interaction term.

3.1 Destabilization of an Isotropic Fermi Surface

As it was shown, the transport equation (2.28) for collective modes can have, under certain circumstances, complex eigenvalues s. These lead to an exponential growth of the fluctuations in Eq. (2.20), implying an instability of the collective mode. The fluctuations thus lead to the evolution of a different ground state. This is the so-called *Pomeranchuk instability*, proposed by Isaak Pomeranchuk [2] in 1958. Such a Pomeranchuk instability may occur in systems with central repulsive interactions which are long- but finite-ranged, as it was concluded by Quintanilla and Schofield [15].

To make this observation more quantitative, one can calculate the free energy for a system where the formerly isotropic Fermi surface is displaced by an angle

dependent amount u (see [13])

$$F - F_0 = V \int_{p_F}^{p_F+u} \frac{\mathrm{d}^d \mathbf{p}}{(2\pi)^d} (\epsilon_{\mathbf{p}} - \mu) + \frac{V^2}{2} \iint_{p_F}^{p_F+u} \frac{\mathrm{d}^d \mathbf{p}}{(2\pi)^d} \frac{\mathrm{d}^d \mathbf{p}'}{(2\pi)^d} f_{\mathbf{p}\mathbf{p}'} \tag{3.1}$$

where Ω and Ω' are again solid angles. With the effective mass $m^\star = p_F/v_F$ the energy relation in the vicinity of the Fermi surface reads

$$\epsilon_{\mathbf{p}} - \mu = \frac{1}{2m^\star}(p^2 - p_F^2) \tag{3.2}$$

Introducing the d-dimensional solid angle Ω of $\mathbf{p}$ and the angle ξ between the directions $\mathbf{p}$ and $\mathbf{p}'$, one can carry out the radial integrations and up to second order in u obtains

$$F - F_0 = V \frac{p_F^d}{2m^\star} \int \frac{\mathrm{d}\Omega}{(2\pi)^d} u(\Omega)^2 + \frac{V^2}{2} \frac{p_F^{2(d-1)}}{(2\pi)^{2d}} \iint \mathrm{d}\Omega\, \mathrm{d}\Omega'\, f(\xi)\, u(\Omega) u(\Omega') \tag{3.3}$$

With the density of states $\nu_0 = V m^\star K_d\, p_F^{d-2}$ this equation simplifies to

$$F - F_0 = \frac{V}{(2\pi)^d} \frac{p_F^d}{2m^\star} \left[\int \mathrm{d}\Omega\, u(\Omega)^2 + \frac{1}{S_d} \iint \mathrm{d}\Omega\, \mathrm{d}\Omega'\, F(\xi)\, u(\Omega) u(\Omega') \right] \tag{3.4}$$

where $F(\xi) = \sum_l F_l P_l(\cos\xi)$ is the reduced interaction energy and must not be confused with the free energy. One may expand the function $u(\Omega)$ in spherical harmonics and use the addition theorem

$$P_l(\cos\xi) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^\star(\Omega) Y_{lm}(\Omega') \tag{3.5}$$

Performing the angular integrations one finally obtains

$$F - F_0 = \frac{V}{(2\pi)^d} \frac{p_F^d}{2m^\star} \sum_l |u_l|^2 (1 + F_l) \qquad \text{for } d = 2, \tag{3.6}$$

$$F - F_0 = \frac{V}{(2\pi)^d} \frac{p_F^d}{2m^\star} \sum_{l,m} |u_{lm}|^2 \left(1 + \frac{F_l}{2l+1}\right) \qquad \text{for } d = 3. \tag{3.7}$$

In order to have a stable isotropic Fermi surface the free energy has to be positive for any possible realization of $\{u_{lm}\}$. This yields the stability criteria

$$\begin{aligned} F_l &\geq -1 & \text{for } d = 2, \\ F_l &\geq -(2l+1) & \text{for } d = 3. \end{aligned} \tag{3.8}$$

For smaller values of the interaction energy the Fermi surface collapses into a symmetry broken phase.

Furthermore, there may occur other, namely *topological* (Lifshitz-like) instabilities of the Fermi sphere which do not imply any symmetry breaking as pointed out in [15]. However, these instabilities are not considered in the following.
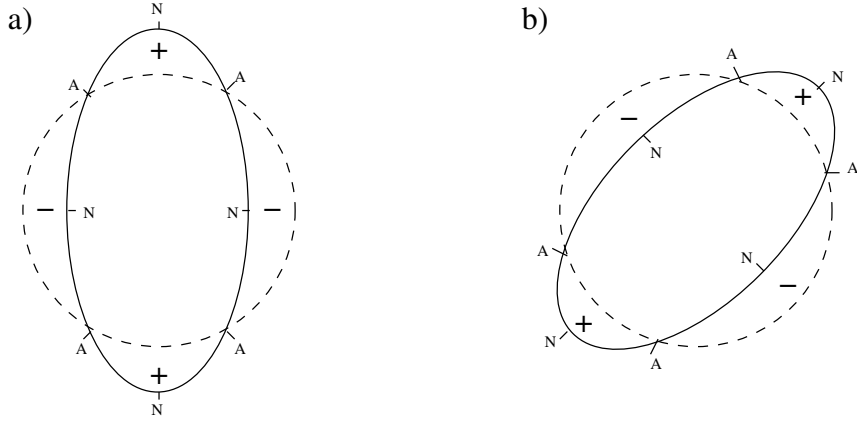
Fig. 3.1: The two different modes of a $l = 2$ Pomeranchuk instability. The formerly isotropic Fermi surface (dashed) gets deformed into a nematic one (straight). The quadrupole symmetry is indicated by plus and minus signs. The letter N denotes the nodal points, while A denotes the antinodal points.

The extension to the spinful case is straightforward for systems which are time-reversal invariant and have a Fermi surface invariant under reflections. All (spin-dependent) quantities like $f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}$ and $u_{\mathbf{p}\sigma}$ can be split into a spin-symmetric and a spin-antisymmetric part. The stability criteria apply to both parts, respectively since they are independent of each other. The best known example of such an instability is the *Stoner instability* which is the one for $l = 0$ in the spin-antisymmetric channel. At a Stoner instability, the Fermi surface of one spin species is enlarged while the Fermi surface of the other one is reduced. This splitting leads to a ferromagnetic transition.

3.2 Landau Damping of the Quadrupolar Mode

In the following, the focus is put on the Pomeranchuk instability with $l = 2$ in $d = 2$ dimensions. Here, the isotropic Fermi surface collapses to an elliptical shape as depicted in Fig. 3.1. There are two linearly independent modes of the excitation, which are rotated about 45° (Note that a 90° rotation yields the first mode again). The positive and negative signs denote the gain or loss of momenta and display the quadrupolar symmetry of the problem. Because of the shape of the resulting Fermi surface such systems are commonly referred to as *nematic Fermi fluids*.

In Sect. 2.4, it was already explained that collective modes are subject to Landau damping. An oscillation with frequency ω and momentum $\mathbf{q}$ decays into a quasiparticle-quasihole excitation with momentum $\mathbf{p} \pm \mathbf{q}/2$ where $\mathbf{p}$ is restricted by the condition $\mathbf{q} \cdot \mathbf{v_p} = \omega$. In the low energy limit, $\omega \to 0$, this condition can only be met for momenta $\mathbf{p}$ where the departure from the isotropic Fermi surface is extremal. As shown in Fig. 3.2, the (position) vector $\mathbf{p}$ and the excitation momentum $\mathbf{q}$ of the

Fig. 3.2: Landau damping is only possible if the momentum **q** of the quasiparticel-quasihole excitation is nearly perpendicular to the momentum **p**. In the low energy regime the momentum **q** has to be nearly tangential to the distorted Fermi surface. For antinodal points (a) both conditions match. At the nodal points (b) the conditions contradict each other and Landau damping can not occur

quasiparticle at the antinodal points can be nearly orthogonal while at the nodal points this is not possible since there is no phase space left. Therefore, one concludes that Landau damping is strong at the antinodal points while it is suppressed at the nodal points.

In contrast, for ferromagnets according to the Stoner instability ($l = 0$), Landau damping is equally important for all possible vectors **q**. This is because the Fermi sphere at a Stoner transition splits into a larger and a smaller Fermi sphere for the two spin species, respectively. However, the Fermi surface is still a sphere and as such the mode is always subject to Landau damping.

3.3 Previous Studies of the Pomeranchuk Instability

In the last decade, Pomeranchuk instabilities, and especially the nematic transition have been widely studied. In a pioneering work Oganesyan et al. [1] explored a transition of two dimensional spinless fermions from an isotropic to a nematic state. Allowing the fermions to interact via a quadrupolar density interaction, they inferred a Landau-Ginzburg-Wilson action for the nematic order parameter. Within a random-phase approximation, they found the single-particle self energy in the symmetry broken phase to leading order to be

$$\text{Im} \, \Sigma \sim |k_x k_y|^{4/3} |\omega|^{2/3} \tag{3.9}$$

where $\mathbf{k} = (k_x, k_y)$ lies on the Fermi surface. The symmetry of the self energy resembles the symmetry of the nematic state. For the directions $(k_x, 0)$ and $(0, k_y)$ along the symmetry axes the self energy reads as

$$\text{Im} \, \Sigma \sim |\omega|^{3/2} \tag{3.10}$$

These results imply a breakdown of Fermi liquid theory where it is $\text{Im} \, \Sigma \sim \omega^2$. Here, the Non-Fermi liquid behavior exists not only at the quantum critical point but in the

whole nematic phase. A non-perturbative calculation performed by Lawler et al.[16] via high dimensional bosonization confirmed this result. At the quantum critical point the self energy obeys the same powerlaw as Eq. (3.9) but with a constant (isotropic) prefactor.

In solids, the underlying lattice breaks rotational symmetry from the outset. Nevertheless, the remaining point group symmetry may spontaneously be broken as well. Since in this case a discrete symmetry is broken, no gapless Goldstone modes exist. Oganesyan et al.[1] argued that this yields at sufficiently low energies to a crossover to a Fermi liquid state.

On a square lattice, the self energy at the quantum critical point and in the disordered phase was calculated ([8] and [3]). At the quantum critical point, dynamical fluctuations of the Fermi surface yield a strongly enhanced decay rate of single-particle excitations . This implies Non-Fermi liquid behavior except at the Brillouin zone diagonals. It was found an expression for the self energy at the Fermi surface ($\mathbf{k} = \mathbf{k}_F$) similar to Eq. (3.9). A prefactor $d_{\mathbf{k}}$ with $d_{x^2-y^2}$ symmetry appears even at the quantum critical point due to the lattice symmetries. At the Brillouin zone diagonals the d-wave symmetry factor vanishes and one obtains a conventional Fermi liquid decay rate. In the low temperature regime away from the quantum critical point, Fermi liquid behavior was recovered below a certain crossover energy.

3.4 Derivation of the Effective Action

The aim of this section is to describe the transition from an isotropic to a nematic state in terms of the order parameter. The subsequent derivation follows the work of Oganesyan et al.[1].

First of all, one has to identify an appropriate order parameter which has to be a real, symmetric and traceless tensor. A natural choice is the quadrupole density which is the expectation value of the quadrupole operator

$$Q = \frac{1}{k_F^2} \bar{\psi} \begin{pmatrix} \partial_x^2 - \partial_y^2 & 2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_y^2 - \partial_x^2 \end{pmatrix} \psi \tag{3.11}$$

The action for this order parameter can be written as

$$S[\bar{\psi}, \psi, Q] = \int \mathrm{d}x \, \mathrm{d}\tau \left[\bar{\psi} \mathcal{G}_0^{-1} \psi + \frac{f_2}{4} \operatorname{Tr} Q^2 \right] \tag{3.12}$$

$\mathcal{G}_0$ is the free fermionic propagator, and f_2 is some interaction constant.

In two dimensions, it is convenient to express the quadrupole density in terms of the Pauli matrices, $Q = \phi_1 \sigma_z + \phi_2 \sigma_x$, where in momentum representation

$$\phi = (\phi_1, \phi_2)^T = \frac{1}{k_F^2} \bar{\psi} \begin{pmatrix} k_x^2 - k_y^2 \\ 2k_xk_y \end{pmatrix} \psi := \bar{\psi} \Delta \psi \tag{3.13}$$

19

Thus, one can rewrite the action in a vectorial form as

$$S[\bar{\psi}, \psi, \phi] = \int \mathrm{d}x \, \mathrm{d}\tau \left[\bar{\psi} \mathcal{G}_0^{-1} \psi + \frac{f_2}{2} \phi^2 \right] \tag{3.14}$$

Nevertheless, one has to keep in mind that the theory remains a tensorial one and the vector components are only the prefactors of the Pauli matrices.

In order to obtain an effective theory for the order parameter one has to eliminate the term quadratic in ϕ, which is quartic in the fermionic fields. This is done by performing a Hubbard-Stratonovich transformation with the field n conjugated to ϕ. Note that n has the same properties as ϕ. The quartic fermionic term is thus replaced by a term quadratic in n

$$\begin{aligned}
S[\bar{\psi}, \psi, n] &= \int \mathrm{d}x \, \mathrm{d}\tau \left[\bar{\psi} \mathcal{G}_0^{-1} \psi + \frac{1}{2f_2} n^2 - in \cdot \phi \right] \\
&= \int \mathrm{d}x \, \mathrm{d}\tau \left[\bar{\psi} \left(\mathcal{G}_0^{-1} - in \cdot \Delta \right) \psi + \frac{1}{2f_2} n^2 \right]
\end{aligned} \tag{3.15}$$

After rescaling $n \to in$ one obtains the partition function $\mathcal{Z}$ as

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}[\psi, \bar{\psi}] \, \mathcal{D}n \, e^{-S[\bar{\psi}, \psi, n]} \\
&= \int \mathcal{D}(\psi, \bar{\psi}) \, \mathcal{D}n \, \exp \left[\int \mathrm{d}x \, \mathrm{d}\tau \left\{ \frac{1}{2f_2} n^2 - \bar{\psi} \left(\mathcal{G}_0^{-1} + n \cdot \Delta \right) \psi \right\} \right]
\end{aligned} \tag{3.16}$$

Then one is able to perform the functional integration over the fermionic degrees of freedom which leads to

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}n \, \det \left[\mathcal{G}_0^{-1} + n \cdot \Delta \right] \exp \left[\int \mathrm{d}x \, \mathrm{d}\tau - \frac{1}{2f_2} n^2 \right] \\
&= \int \mathcal{D}n \, \exp \left[\frac{1}{2f_2} \int \mathrm{d}x \, \mathrm{d}\tau \, n^2 + \mathrm{Tr} \left\{ \ln \left(\mathcal{G}_0^{-1} + n \cdot \Delta \right) \right\} \right]
\end{aligned} \tag{3.17}$$

where in the second line the common identity $\ln(\det A) = \mathrm{Tr}(\ln A)$ for any arbitrary non-singular operator A was used. Finally, one expands the logarithm to second order.

$$\begin{aligned}
\mathrm{Tr} \left\{ \ln \left(\mathcal{G}_0^{-1} + n \cdot \Delta \right) \right\} &= \mathrm{Tr} \ln \mathcal{G}_0^{-1} + \mathrm{Tr} \left\{ \ln \left(1 + \mathcal{G}_0 n \cdot \Delta \right) \right\} \tag{3.18} \\
&= -S_0 + \mathrm{Tr} \left\{ \mathcal{G}_0 n \cdot \Delta \right\} - \frac{1}{2} \mathrm{Tr} \left\{ \left(\mathcal{G}_0 n \cdot \Delta \right) \left(\mathcal{G}_0 n \cdot \Delta \right) \right\} + \dots
\end{aligned}$$

where S_0 is the constant fermionic part of the action which is dropped in the following.

In App. A, the detailed calculation of these traces is presented explicitly. Here, only the resulting effective action is given, which to quadratic order in momentum representation reads

$$S_{\mathrm{eff}}[n] = S_0 + \frac{1}{2} \frac{T}{L^d} \sum_q n_q \, g_0(q)^{-1} n_{-q} \tag{3.19}$$

20

The summation index $q = (\mathbf{q}, \omega)$ is the *three-momentum*, comprising momentum and frequency.

In the frame of reference in which the momentum $\mathbf{q}$ is parallel to the x-axis, the matrix g_0^{-1} takes the diagonal form

$$g_0^{-1} = \begin{pmatrix} g_3^{-1} & \\ & g_2^{-1} \end{pmatrix} = \begin{pmatrix} r_0 + q^2 + \gamma \frac{|\omega|}{q} & \\ & r_0 + q^2 + \frac{\omega^2}{q^2} \end{pmatrix} \tag{3.20}$$

with rescaled momenta and frequencies and $\gamma = 1/\sqrt{2}$. In Ginzburg-Landau theories the phase transition takes place at vanishing mass $r_0 = 0$. Here, the mass is equal to $r_0 = \left(-\frac{1}{2} - \frac{1}{\nu f_2}\right)$ and therefore vanishes at $\nu f_2 = -2$. At this point, the isotropic Fermi surface gets unstable in the quadrupolar channel, and comparison to Eq. (3.8) leads to the identification $\nu f_2 = -2F_2$ where F_2 is the reduced Landau parameter.

Since the order parameter explicitly breaks spatial rotational symmetry (instead of an intrinsic symmetry like spin rotation) the vector n_q is in this frame of reference angle-dependent. For practical purposes a more convenient frame of reference is the one in which the order parameter is kept rotationally invariant, and the inverse propagator becomes angle-dependent. Since the quadrupolar momentum tensor is invariant under a rotation by π, the same applies to the momentum dependence of n_q. Therefore, the transformation matrix for n is given by

$$U(\hat{\mathbf{q}}) = \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ -\sin(2\phi) & \cos(2\phi) \end{pmatrix} \tag{3.21}$$

where ϕ is the angle between $\hat{\mathbf{q}}$ and the x-axis. The rotation of n_q by an angle of 2ϕ resembles the fact that the order parameter of a nematic phase is a director, rather than a vector, which is invariant under rotation by π. With this transformation one gets

$$n_q^T g_0(q)^{-1} n_{-q} = \tilde{n}_q^T U(\hat{\mathbf{q}})^T g_0(q)^{-1} U(-\hat{\mathbf{q}}) \tilde{n}_{-q} = \tilde{n}_q^T g(q)^{-1} \tilde{n}_{-q} \tag{3.22}$$

Here, by $\tilde{n}_q$ the new rotational invariant fields are denoted. However, the tilde is dropped in the following. The matrix $g(q)^{-1}$ is explicitly given by

$$g^{-1} = \begin{pmatrix} g_2^{-1} \sin^2(2\phi) + g_3^{-1} \cos^2(2\phi) & (g_3^{-1} - g_2^{-1}) \sin(2\phi) \cos(2\phi) \\ (g_3^{-1} - g_2^{-1}) \sin(2\phi) \cos(2\phi) & g_2^{-1} \cos^2(2\phi) + g_3^{-1} \sin^2(2\phi) \end{pmatrix} \tag{3.23}$$

One may also rewrite the propagator by separating g^{-1} into the contribution of both modes as $g(q)^{-1} = g_2^{-1} U_2(\hat{\mathbf{q}}) + g_3^{-1} U_3(\hat{\mathbf{q}})$ where the matrices $U_2(\hat{\mathbf{q}})$ and $U_3(\hat{\mathbf{q}})$ can be read from Eq. (3.23).

There are two peculiarities concerning the action. First of all, both modes, g_2 and g_3, have the same mass r_0. This implies that they become critical at the same point in parameter space. Renormalizations of the mass due to interactions do not

modify this fact since it is protected by symmetry. In contrast, the prefactors of the momentum term are not equal because of symmetry constrains. Thus, higher order terms give rise to different prefactors, which, however, is only of little qualitative importance.

Second, the two modes have completely different dynamics. While the g_3 mode is subject to Landau damping, as indicated by the term $|\omega|/q$, the mode g_2 is undamped and therefore ballistic. In particular, both modes show even a different scaling behavior. At criticality, one has to compare the dynamical part with the q^2 term. Obviously, ω scales like q^z with $z = 2$ for the g_2 mode and with $z = 3$ for the g_3 mode. These two different dynamical exponents have a twofold consequence.

Concerning different temperature regimes, the two modes have different importance. At $T = 0$, the g_2 mode has the effective dimension $d + z = 4$ and is therefore at the upper critical dimension, whereas the g_3 mode is above it. This implies that in a perturbative approach the g_2 mode generates logarithmic IR-divergences. On the other hand, at finite temperatures thermodynamical properties are governed by the mode with the larger dynamical exponent. For example the specific heat at low temperatures usually scales as $T^{d/z}$.

Moreover, the logarithmic divergences prevent analyzing the model by means of plain perturbation theory to any order. These divergences have to be summed up which is usually done by renormalization group techniques. But as it is specified below, it is far from obviouse how RG works in the presence of two dynamical exponents. This is the major problem in order to solve the problem in a perturbative way.

In the work of Oganesyan et al. [1] the authors state that the g_2 mode plays no role for the critical theory. They argue, that the g_2 mode has a characteristic energy of $\omega_2 \sim q^2$ while the g_3 mode has a characteristic energy of $\omega_3 \sim q^3$. Thus, the energy of the g_3 mode is lower and therefor dominating the system. For this reason they justify a $z = 3$ scaling, for which the term $\omega^2/q^2 \sim q^4$ is, of course, irrelevant. Furthermore, the conclusion is drawn that for a theory with $z = 3$ in two spatial dimensions interaction terms of order $\mathbf{n}^4$ are according to Hertz [17] and Millis [18] irrelevant.

However, the choice of scaling with $z = 2$ is not prohibited, in which case a quartic interaction is only marginal. Additionally, the arising logarithms of the g_2 mode may play an important role. As it will be shown, the g_2 mode, indeed, governs the zero temperature correlation length of the system and has also a high impact on its finite temperature properties. Notably, in lattices there exists no mode with $z = 2$ but Landau damping is suppressed only at the Brillouin zone diagonals so that the results of $z = 3$ scaling are correct.

Taking interactions into account, one formally obtains them by expanding the logarithm of Eq. (3.18) to higher orders. However, the cubic term vanishes due

to the fact that the original order parameter is a traceless tensor. (Note that in dimensions larger than two this statement does not hold, and therefore a cubic term generically arises. This implies that the transition is then of first order.) In two dimensions the lowest-order interaction is the quartic term which by symmetry is constrained to the form

$$S_{\text{int}}[n] = \frac{u_0}{4!} \left(\frac{T}{L^d} \right)^3 \sum_{\substack{k_1, k_2 \\ k_3, k_4}} (n_{k_1} n_{k_2}) (n_{k_3} n_{k_4}) \, \delta_{k_1 + k_2 + k_3 + k_4} \tag{3.24}$$

where u_0 is a coupling constant. The exact form of u_0 does not play a role. Notably, there is only one coupling constant u_0 which accounts for all three possible interaction terms n_2^4, $n_2^2 n_3^2$ and n_3^4 (here the subscripts indicate the different modes, not the momentum index). This fact plays an important role in the renormalization of the coupling constants.

Collecting everything, one obtains the effective quartic action of the order parameter field n as

$$S[n] = \frac{1}{2} \frac{T}{L^d} \sum_q n_q g_q^{-1} n_{-q} + \frac{u_0}{4!} \left(\frac{T}{L^d} \right)^3 \sum_{\substack{k_1, k_2 \\ k_3, k_4}} (n_{k_1} n_{k_2}) (n_{k_3} n_{k_4}) \, \delta_{k_1 + k_2 + k_3 + k_4} \tag{3.25}$$

This action is the starting point of the further analysis of the quadrupolar Pomeranchuk instability.

Chapter 4

Analysis at Zero Temperature

In order to analyze the effective action one considers the quartic interaction term as a small perturbation. As mentioned before perturbation theory leads to logarithmic divergences due to the mode g_2 being at the upper critical dimension. In order to sum up all these divergences a powerful tool is the so-called renormalization group techniques (RG). However, as the model has two coupled modes with different dynamics there is no established RG scheme.

In this chapter, a RG scheme for the problem at hand is proposed and applied. In the first section, a brief review of the general scheme of momentum-shell RG with a single critical dynamical exponent z is given. In the second section, the RG scheme is described which is applied to the problem under consideration. Within this scheme the calculation of the β functions at zero temperature for the mass r and the vertex u are calculated. Hence, one can deduce the correlation length ξ. The result states that this theory is neither in the Ising universality class nor in the universality class of the XY model.

4.1   General Idea of Renormalization Group Methods

Every theory is defined by the action

$$S[\phi] = \sum_i g_i \hat{O}_i[\phi] \tag{4.1}$$

Here, ϕ is in general a multi-component field over which the functional integral of the partition function extends, and $\{g_i\}$ is a set of coupling constants connected to a certain set of operators $\{\hat{O}_i\}$.

The RG method consists of two stages. In the first step one divides the field, given in momentum space representation $\{\phi_k\}$, into a high-momentum sector

$$\phi_{\mathrm{h}}(k) = \begin{cases} \phi(k) & \text{for } k \in \mathcal{S} \\ 0 & \text{else} \end{cases} \tag{4.2}$$
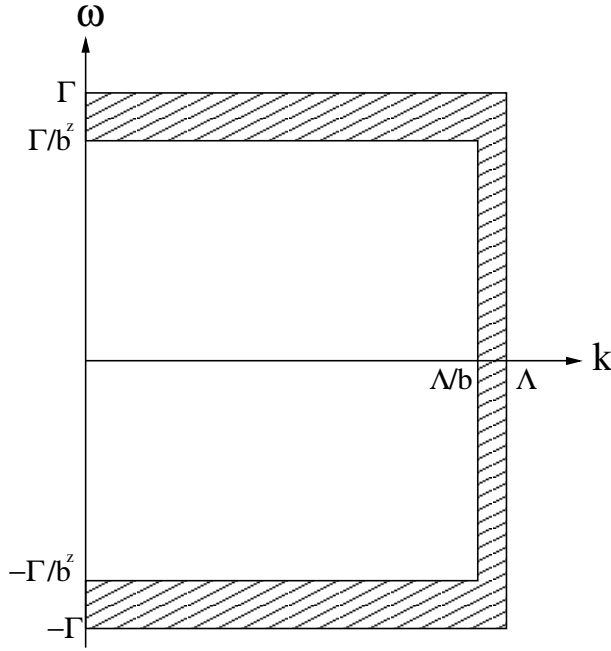
Fig. 4.1:

Partition of the k-ω space into the low-momentum sector (white) and the high-momentum sector $\mathcal{S}$ (shaded). The factor $b = 1 + \epsilon$ is a number close to unity.

and a low-momentum sector

$$\phi_\mathrm{l}(k) = \begin{cases} 0 & \text{for } k \in \mathcal{S} \\ \phi(k) & \text{else} \end{cases} \tag{4.3}$$

where $\mathcal{S}$ is the shaded area depicted in Fig. 4.1.

The action can be divided into three parts

$$S[\phi] = S_\mathrm{h}[\phi_\mathrm{h}] + S_\mathrm{l}[\phi_\mathrm{l}] + S_\mathrm{int}[\phi_\mathrm{h}, \phi_\mathrm{l}] \tag{4.4}$$

Integrating out the high momentum sector ϕ_h, one obtains an action depending only on the low-momentum sector of ϕ

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-S[\phi]} = \int \mathcal{D}\phi_\mathrm{h} \mathcal{D}\phi_\mathrm{l} \, e^{-S_\mathrm{h}[\phi_\mathrm{h}] - S_\mathrm{l}[\phi_\mathrm{l}] - S_\mathrm{int}[\phi_\mathrm{h}, \phi_\mathrm{l}]}$$
$$= \int \mathcal{D}\phi_\mathrm{l} \, e^{-S'[\phi_\mathrm{l}]} \tag{4.5}$$

The action S' is formally defined by

$$e^{-S'[\phi_\mathrm{l}]} = e^{-S_\mathrm{l}[\phi_\mathrm{l}]} \left\langle e^{-S_\mathrm{int}[\phi_\mathrm{h}, \phi_\mathrm{l}]} \right\rangle_h \tag{4.6}$$

where the bracket $\langle \dots \rangle_h = \int \mathcal{D}\phi_\mathrm{h} \, \dots e^{-S_\mathrm{h}[\phi_\mathrm{h}]}$ denotes an average over the high-momentum field only. Writing the resulting action in terms of coupling constants and operators one gets

$$S'[\phi_\mathrm{l}] = \sum_i g'_i \hat{O}'_i[\phi_\mathrm{l}] \tag{4.7}$$

The coupling constants g'_i are in general different from the g_i appearing in the original action. Moreover, the integration over the fast fields can generate even new

operators, which are not present in the original action. In this case, one has to go a step back and add such operators to the original action, e.g. by taking further effects into account. This step is repeatedly done until one arrives at a system which is closed under RG transformation and thus called *renormalizable*.

However, the theories are still not yet comparable. While the original fields fluctuate in a momentum scale $[0, \Lambda]$, the low-momentum fields fluctuate only in a momentum scale $[0, \Lambda/b]$; analogues applies for the frequencies. In order to solve this problem one rescales momenta according to $k \to bk$ as well as frequencies according to $\omega \to b^z \omega$ where the so called *dynamical critical exponent z* depends on the effective dispersion relation. Finally, one has the freedom of rescaling the field ϕ arbitrarily, since it is an integration variable. In general, one rescales the ϕ field such that the leading order gradient term remains invariant. The resulting action

$$S'[\phi] = \sum_i g'_i \hat{O}_i[\phi] \tag{4.8}$$

differs from the original one only by the values of the new coupling constants g'_i. Therefore, the full information about the RG step is encapsulated in the mapping $g_i \to g'_i \equiv g_i(b)$. For infinitesimal $l = \ln b$ this change is displayed most conveniently by the so-called *β functions*

$$\beta_i(g_i) = \frac{dg_i(b)}{d \ln b}\bigg|_{b=1} := \lim_{l \to 0} \frac{g_i(b) - g_i}{l} \tag{4.9}$$

The β functions describe the flow of the coupling constants under subsequent RG transformation.

A special role play the *fixed points g^0* of the RG flow for which all $\beta_i(g_i^0) = 0$. The RG flow can be interpreted as looking at the system on larger and larger scales, averaging over fast fluctuations or small lengths. At a fixed point, the coupling constants do not change under this flow which implies that the system at the larger scale looks the same as at the smaller scales. On the other hand, every model has an intrinsic length scale ξ determining the spatial correlations of field fluctuations. Both observations agree only if the system is either in a state where ξ vanishes, or in an totally ordered phase where the correlation length diverges. However, a diverging correlation length is a hallmark of a phase transition and, thus, the phase diagram can be qualitatively obtained by means of RG analysis

There are different possibilities for the coupling constants in the vicinity of a fixed point. *Relevant* coupling constants are those that under the RG transformation are exponentially driven away from the fixed point. This means that the difference to the value at the fixed point, $\Delta g_i = g_i - g_i^0$, increases. In contrast, *irrelevant* coupling constants are those for which Δg_i exponentially decrease. Coupling constants which increase or decrease slower than exponentially are said to be *marginally relevant* or *marginally irrelevant*. Of course, there are further specifications of the type of a coupling constant which are not discussed here.

With these definitions one can classify the fixed points. If there is a whole neighborhood in coupling constant space in which all coupling constants are (marginally) irrelevant, the fixed point is said to be *stable*. A small deviation from the fixed point will at least asymptotically flow back towards the fixed point. These stable fixed points resemble the stable phases. On the other hand, in the neighborhood of an *unstable* fixed points all coupling constants, taken into consideration, are (marginally) relevant. These fixed points do not correspond to any state of matter. The third class of fixed points have, of course, relevant as well as irrelevant coupling constants in their neighborhood. These fixed points can be associated with phase transitions. A deviation in the direction of an irrelevant coupling constant is driven back to the fixed point. In contrast, the smallest deviation in any other direction will grow and drive the system towards another fixed point corresponding to another phase.

Although there is much more to say about the renormalization group method, this discussion concludes the current section. For a further introduction into renormalization group methods applied to quantum critical phenomena there are, of course, the articles of Hertz [17] and Millis [18] as well as various textbooks, e.g. [19] and [20].

4.2 Renormalization Group Method at Zero Temperature

The model under consideration has the action (3.25) with the propagator (3.23). One has to recall that the mode g_2 has, at zero temperature, an effective dimension $d + z = 4$, which is the upper critical dimension. In a perturbative expansion this leads to logarithmic divergences in any given order. The RG method provides a tool to sum up all these divergencies to a convergent β function.

However, following the scheme presented above yields a serious problem. After integrating out the fast fluctuating fields one has to rescale both momenta, $q \rightarrow bq$, and frequencies, $\omega \rightarrow b^z \omega$. Here, the effective theory contains two different modes with different dynamics and especially different dynamical exponents. Therefore, it is not obvious how to perform the rescaling procedure for frequencies.

First, one may recall the influence of both modes. While the g_2 mode gives rise to logarithmic IR divergences, the g_3 mode is above the upper critical dimension and therefore yields IR-convergent corrections. Because of this, it is reasonable to focus on the g_2 mode and rescale according to $z = 2$. This, of course, implies that after rescaling momenta and frequencies the g_3 mode is no longer invariant but obtains an additional factor of b according to $\gamma_0 |\omega|/q \rightarrow b\gamma_0 |\omega|/q$. To compare the low energy theory with the full theory, this factor can be absorbed into a rescaling of γ_0. Although in the original action γ_0 is only a plain number it now becomes a running

a)

b)

Fig. 4.2: Diagrammatic representation of the corrections contributing in one-loop order. Each line has a momentum and a frequency label as well as a label $i = 1, 2$ denoting the component of the order parameter

"coupling constant" obeying the RG equation $\gamma(b) = b\gamma_0$. The fact that γ increases under the RG flow again weakens the importance of the g_3 mode which is exactly taken into account. Since the corresponding propagator for large b now scales as $\sim \gamma^{-1} q/|\omega|$ it is suppressed as γ increases.

Within this scheme the small momentum action can be rescaled and the β function for mass $r(b)$ and coupling constant $u(b)$ are obtained. This is done within a loop expansion to first order. In the following the main steps are presented while a more detailed calculation is given in App. B.

The separation into small and large momentum fields leads to the interaction

$$S_{\mathrm{int}}[n_{\mathrm{l}}, n_{\mathrm{h}}] = \frac{u_0}{4!} \left(\frac{T}{L^d}\right)^3 \left[2 \sum_{q_1, q_2, k_3} n_{q_1}^{\mathrm{i}} n_{q_2}^{\mathrm{i}} n_{k_3}^{\mathrm{j}} n_{-q_1-q_2-k_3}^{\mathrm{j}} + 4 \sum_{q_1, k_2, q_3} n_{q_1}^{\mathrm{i}} n_{k_2}^{\mathrm{i}} n_{q_3}^{\mathrm{j}} n_{-q_1-k_2-q_3}^{\mathrm{j}} \right]$$

(4.10)

where the convention $q \in \mathcal{S}$ and $k \notin \mathcal{S}$ is used and $\mathcal{S}$ was defined in Fig. 4.1. The low-momentum action looks like the original one while the fast action to one-loop order consists of the quadratic term only.

Now one has to calculate the average $\langle e^{-S_{\mathrm{int}}[n_{\mathrm{h}}, n_{\mathrm{l}}]} \rangle_h = \int \mathcal{D}\phi_{\mathrm{h}} \, e^{-S_{\mathrm{int}}[n_{\mathrm{h}}, n_{\mathrm{l}}]} e^{-S_{\mathrm{h}}[\mathrm{h}]}$. Within a one-loop order approximation, one can expand the exponential and re-exponentiate it after averaging which leads to

$$\left\langle e^{-S_{\mathrm{int}}[n_{\mathrm{h}}, n_{\mathrm{l}}]} \right\rangle_{\mathrm{h}} = \exp\left[- \langle S_{\mathrm{int}}[n_{\mathrm{h}}, n_{\mathrm{l}}] \rangle_{\mathrm{h}} + \frac{1}{2} \langle S_{\mathrm{int}}[n_{\mathrm{h}}, n_{\mathrm{l}}]^2 \rangle_{\mathrm{h}}^c \right]$$

(4.11)

where the superscript c denotes that only connected diagrams are taken into account.

The diagrammatic representations of the two terms are depicted in Fig. 4.2. The first one leads to a correction of the mass term, δr, while the second one renormalizes the quartic term and therefore gives rise to a vertex correction δu.

29

4.2.1 Mass Correction

First, the mass correction is calculated, which corresponds to the diagram depicted in Fig. 4.2 (a) with zero external momentum. The complete correction is given by

$$\langle S_{\text{int}}[n_{\text{h}}, n_{\text{l}}] \rangle_{\text{h}} = 2\frac{u}{4!} \left(\frac{T}{L^d}\right)^3 \sum_{k_1, k_2, k_3} \langle n_{k_1}^{\text{i}}\, n_{k_2}^{\text{i}} \rangle_{\text{h}}\, n_{k_3}^{\text{j}}\, n_{-k_1-k_2-k_3}^{\text{j}}$$

$$+ 4\frac{u}{4!} \left(\frac{T}{L^d}\right)^3 \sum_{k_1, k_2, k_3} n_{k_1}^{\text{i}} \left\langle n_{k_2}^{\text{i}} n_{k_3}^{\text{j}} \right\rangle_{\text{h}} n_{-k_1-k_2-k_3}^{\text{j}} \qquad (4.12)$$

where the sum over $\{i, j\}$ is implicit. Since the propagator is diagonal in momentum space it is $\langle n_k^{\text{i}} n_{k'}^{\text{j}} \rangle \propto \delta_{k,-k'}$. Moreover, one can show that after angular averaging

$$\sum_k \left\langle n_k^{\text{i}}\, n_{-k}^{\text{j}} \right\rangle = \delta_{\text{i,j}} \sum_k \frac{g_2 + g_3}{2} \qquad (4.13)$$

which after summation over $\{i, j\}$ leads to the mass correction

$$\delta r = \frac{2(N+2)u}{4!} \int_{\mathcal{S}} \mathrm{d}k\, k \frac{\mathrm{d}\phi}{(2\pi)^2} \frac{\mathrm{d}\omega}{2\pi} \left(\frac{1}{r + k^2 + \gamma_0|\omega|/k} + \frac{1}{r + k^2 + (\omega/k)^2}\right) \qquad (4.14)$$

In general N is the number of components of the field which in the problem at hand is $N = 2$. This factor originates in the free summation over i while contracting $\langle n_k^i n_{k'}^i \rangle$. The integration runs over the high momentum/frequency shell $\mathcal{S}$ depicted in Fig. 4.1 since the average is taken over the fast fields only. Note the factor of 2 due to the factor of $\frac{1}{2}$ in front of the quadratic term of the action.

To obtain the β function, one has to take the derivative with respect to $\ln b$, which appears in the boundaries of the integrals. This means that one has only to perform either of the k- and ω-integration while the other variable is kept fixed to the value of the cutoff.

$$\frac{\partial(\delta r)}{\partial \ln b} = \frac{K_2 u}{6} \left\{ \frac{\Lambda}{b} \int_{-\Gamma}^{\Gamma} \frac{d\omega}{2\pi} \left(\frac{\Lambda}{r + \Lambda^2 + \gamma_0|\omega|/\Lambda} + \frac{\Lambda}{r + \Lambda^2 + (\omega/\Lambda)^2}\right) \right.$$

$$\left. + 2z\Gamma b^{-z} \int_0^{\Lambda} \mathrm{d}k k \left(\frac{k}{r + k^2 + \gamma_0\Gamma/k} + \frac{k}{r + k^2 + (\Gamma/k)^2}\right) \right\} \qquad (4.15)$$

As before, $K_2 = \int \mathrm{d}\phi/(2\pi)^2$ denotes the averaged volume of the 2-dimensional sphere. Expansion to first order in the control parameter r yields ($z = 2$):

$$\frac{\partial(\delta r)}{\partial \ln b} = \frac{K_2 u}{6\pi} \left\{ 2\Lambda^2 \arctan\left(\frac{\Gamma}{\Lambda^2}\right) + \Gamma \ln\left(1 + \frac{\Lambda^4}{\Gamma^2}\right) + 2\frac{\Lambda^3}{\gamma} \ln\left(1 + \frac{\gamma\Gamma}{\Lambda^3}\right) + \frac{4}{3}\Gamma \ln\left(1 + \frac{\Lambda^3}{\gamma\Gamma}\right) \right.$$

$$- r\frac{\pi}{2} - \frac{2}{3}\frac{r\Gamma}{(\gamma\Gamma)^{2/3}} \left[\frac{\sqrt{3}}{9}\pi + \Lambda\frac{(\gamma\Gamma)^{2/3}}{\Lambda^3 + \gamma\Gamma} + \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\Lambda - (\gamma\Gamma)^{1/3}}{\sqrt{3}(\gamma\Gamma)^{1/3}}\right)\right.$$

$$\left.\left.+ \frac{1}{3}\ln\left(\frac{\left(\Lambda + (\gamma\Gamma)^{1/3}\right)^2}{(\gamma\Gamma)^{2/3} - \Lambda(\gamma\Gamma)^{1/3} + \Lambda^2}\right)\right]\right\} \qquad (4.16)$$

Note that the terms in the first line are independent of r. Therefore these terms can be absorbed into a renormalization of the bare parameter r_0. This is a renormalization of the *starting point* of the RG flow.

To obtain the asymptotic RG flow, one has to identify the leading order behavior. The terms within the rectangular brackets, which originate from the g_3 mode, are suppressed for large b by virtue of the growth of γ. Their influence on the asymptotic flow due to the small b behavior can be captured in another shift of the bare coupling constant. Asymptotically the β function of r relies on the contribution of the g_2 mode only.

Together with the engineering dimension of r one ends up with the asymptotic RG equation

$$\frac{\partial r}{\partial \ln b} = 2r - \frac{uK_2}{12}r \tag{4.17}$$

with $K_2 = \frac{1}{2\pi}$.

4.2.2 Vertex Correction

In the second step the vertex correction, δu, is calculated which corresponds to the diagram depicted in Fig. 4.2 (b). More specific, the action gets the additional terms

$$\frac{1}{2}\left\langle S_{\text{int}}[n_{\text{h}}, n_{\text{l}}]^2 \right\rangle_{\text{h}}^c = \frac{1}{2}\left(\frac{u}{4!}\right)^2\left(\frac{T}{L^d}\right)^4 \left[8\sum_{k_1,k_2,k} n_{k_1}^{\text{i}} n_{-k_1+k}^{\text{i}} \left(\sum_q \left\langle n_q^{\text{j}} n_{-q}^{\text{k}}\right\rangle \left\langle n_{q-k}^{\text{j}} n_{k-q}^{\text{k}}\right\rangle\right) n_{k_2}^{\text{l}} n_{-k_2-k}^{\text{l}} \right.$$

$$+ 32 \sum_{k_1,k_2,k} n_{k_1}^{\text{i}} \, n_{-k_1+k}^{\text{j}} \left(\sum_q \left\langle n_q^{\text{i}} \, n_{-q}^{\text{k}}\right\rangle \left\langle n_{q-k}^{\text{j}} \, n_{k-q}^{\text{k}}\right\rangle\right) n_{k_2}^{\text{l}} \, n_{-k_2-k}^{\text{l}}$$

$$\left. + 32 \sum_{k_1,k_2,k} n_{k_1}^{\text{i}} \, n_{-k_1+k}^{\text{j}} \left(\sum_q \left\langle n_q^{\text{i}} \, n_{-q}^{\text{k}}\right\rangle \left\langle n_{q-k}^{\text{j}} \, n_{k-q}^{\text{l}}\right\rangle\right) n_{k_2}^{\text{k}} \, n_{-k_2-k}^{\text{l}} \right] \tag{4.18}$$

where again the sum over {i, j, k, l} is implicit. With the assumption that the vertex correction does not vary much with k, one can approximate it by its value at momentum $k = 0$. Hence, one ends up with

$$\delta u = \frac{1}{2}\frac{u^2 K_2}{4!}\left[\int_{\mathcal{S}} \mathrm{d}q\, q \frac{\mathrm{d}\omega}{2\pi} 4(N+7)\left(g_2(q,\omega)^2 + g_3(q,\omega)^2\right) + 8\,g_2(q,\omega)g_3(q,\omega)\right] \tag{4.19}$$

where the factor N, the number of components of the order parameter, arises from the free summation over i in the first term, and the integration runs over the same shell, $\mathcal{S}$, as the mass correction.

Since the further calculation of the RG equation for δu is conceptual equivalent to the one for δr, it is not presented here in detail. To summarize, the three terms g_2^2, g_3^2 and $g_2\,g_3$ yield a renormalization of the bare coupling constant u_0 in the

first place. Asymptotically, the flow is governed by the g_2^2 contribution only, and to second order in u reads as

$$\frac{\partial u}{\partial \ln b} = (4 - d - z)u - \frac{3K_2}{16}u^2 \tag{4.20}$$

The detailed calculation which yields this result is carried out in App. B.

4.2.3 Solution of the RG-Equations

Finally, one obtains a set of coupled differential equations for the asymptotic flow of the mass r and the coupling u:

$$\frac{\partial r}{\partial \ln b} = 2r(b) - \frac{K_2}{12}u(b)r(b) \tag{4.21}$$

$$\frac{\partial u}{\partial \ln b} = (4 - d - z)u(b) - \frac{3K_2}{16}u^2(b) \tag{4.22}$$

This set of coupled differential equations can be solved analytically which yields ($d = 2$ and $z = 2$)

$$r(b) = rb^2 \left(\frac{u(b)}{u}\right)^{4/9} \tag{4.23}$$

$$u(b) = \frac{c}{\ln(be^{c/u})} \tag{4.24}$$

with $c = 16/(3K_2)$. The constants r and u denote the renormalized bare coupling constants, which are different from the microscopic ones r_0 and u_0. Moreover, one has to keep in mind that γ obeys the RG equation

$$\gamma(b) = \gamma_0 \, b \tag{4.25}$$

From these results one can calculate the correlation length ξ. To this end, one has to estimate at which value of b the RG flow has to be stopped. This is the case when the momentum cutoff Λ is no longer the largest scale of the problem or, in other words, when $r(b_\star) = \Lambda^2$. Neglecting the logarithmically small factor $(u(b)/u)^{4/9}$ this happens for $b_\star = \Lambda/\sqrt{r}$. Using the scaling relation

$$\xi^{-2} \sim R(b_\star) = r(b_\star)b_\star^{-2} \tag{4.26}$$

one obtains for the correlation length

$$\xi^{-2} \sim r \left(\frac{c}{(\ln(\Lambda^2 e^{c/u}/r)}\right)^{4/9} \sim \frac{r}{(\ln(c_1/r))^{4/9}} \tag{4.27}$$

Thus, in comparison to the naive guess, $\xi^{-2} \sim r$, the correlation length is enhanced by a factor of $\ln(1/r)^{2/9}$.

The main feature of this solution is the exponent $4/9$. Right from the beginning one might have guessed that a theory of this type is in the Ising universality class.

Although one has a two component order parameter, both components have the same mass and interaction constant. From the asymptotic calculation one infers, that only one mode, namely the mode g_2, contributes to the corrections, and thus one could expect an Ising-like behavior with an exponent of 3/9. On the other hand, this argument may be oversimplified since the order parameter still consists of two components. Because of that, one may also expect the exponent of the XY model which is 4/10. However, the above calculation yields an exponent which is even larger than the one of the XY model, cf. Eq.4.27.

The reason for this is the interplay of the both facts that on the one hand the order parameter has two components, but on the other hand only one of the corresponding modes is relevant and thereby renormalizes the interaction of both modes. To be specific, the corrections have the general form

$$\delta r = \frac{2uK_d}{4!}(N+2)\int (g_2+g_3) \tag{4.28}$$

$$\delta u = \frac{2u^2K_d}{4!}\left[(N+7)\int (g_2^2+g_3^2) + \int 2g_2g_3\right] \tag{4.29}$$

The exponent is obtained by dividing the combinatorial prefactors of δr and δu. For the $O(N)$ model there exists only one mode implying $g_2 = g_3 \equiv g$. This leads to $\delta r = \frac{4uK_d}{4!}(N+2)\int g$ and $\delta u = \frac{4u^2K_d}{4!}(N+8)\int g$ and an exponent of

$$\frac{4(N+2)/4!}{4(N+8)/4!} = \frac{N+2}{N+8} \tag{4.30}$$

For the XY model one has to set $N = 2$ yielding an exponent of $\frac{4}{10}$ while the Ising model has only a single component order parameter implying an exponent of $\frac{3}{9}$. This result can be generalized to arbitrary N and marks the universality class of $O(n)$ models.

However, in this problem the g_2 mode and the g_3 mode are not equal but one of them does not generate logarithms and, therefore, was irrelevant ($g_3 = 0$). Since we, nevertheless, have a two-component order parameter (implying $N = 2$) the corrections are $\delta r = \frac{uK_d}{12}(N+2)\int g_2$ and $\delta u/u = \frac{uK_d}{12}(N+7)\int g_2^2$ and the exponent reads

$$\frac{(N+2)uK_d/12}{(N+7)uK_d/12} = \frac{N+2}{N+7} = \frac{4}{9} \tag{4.31}$$

As a consequence, the Pomeranchuk problem is neither in the universality class of the XY model nor in the Ising universality class.

4.3 Summary

In this chapter the usual RG scheme was adjusted such that it is capable to capture the two different dynamical exponents. This was done by treating the constant γ_0 in the $z = 3$ term as a flowing prefactor $\gamma(b)$.

With this RG scheme the Pomeranchuk model was analyzed. In particular the interaction renormalization of the mass, r, and the vertex u are calculated. It turned out that the asymptotic RG flow of these two coupling constants is determined by the g_2 mode only. This observation contradicts the arguments given by Oganesyan et al. [1] yielding that the g_2 mode is irrelevant for the critical behavior. At least in the case of the correlation length, the naive scaling arguments fail and therefore this conclusion is wrong.

The correlation length was deduced from the RG-flow of the mass. At zero temperature it is $\xi^{-2} \sim r \ln^{-4/9}(c/r)$. This result indicates that the model is neither in the Ising nor in the XY universality class.

Chapter 5

Finite Temperatures

In the previous chapter, the quadrupolar Pomeranchuk instability was explored at zero temperature where a quantum phase transition takes place. At the critical point where the renormalized mass vanishes, the system undergoes a phase transition from an isotropic state to a nematic ordered state. However, this is a somehow academic problem, because of the unattainability of the absolute zero point. At finite temperatures and in two dimensions an ordered phase with a diverging correlation length is prohibited due to Mermin-Wagner theorem. Nevertheless, above the quantum critical point thermal fluctuations result in excitations of the quantum critical ground state. One can therefore expect some interesting physics in the quantum critical regime.

In this chapter, two different techniques are applied to analyze the system in the quantum critical regime. In the first section a perturbative approach based on a rather general method called dimensional reduction is used. However, as one will see below, this method breaks down for the problem under consideration.

Therefore, in the second section, the renormalization group scheme developed in the previous chapter is applied at finite temperatures. Finally, the finite correlation length in the quantum critical region is obtained to leading order in the temperature, which shows universal behavior.

5.1   RG-Improved Perturbation Theory

In this section, the first approach to the system at finite temperatures is made. At zero temperature, divergences emerge due to the effective dimension $d + z = 4$ of the g_2 mode. At high temperatures the system non-zero Matsubara modes are gapped. As pointed out below, one may think of integrating them out perturbatively to obtain an effective model for the zero mode with the dimension $d = 2$. In the first section, general considerations for both modes at finite temperatures are made, respectively. Thereafter, the scheme of dimensional reduction is applied to the model

at hand and shown to fail due to the coupling of both modes.

5.1.1 General Idea

As pointed out, one can divide the momentum space at finite but low temperatures into different regimes. Since the zero temperature renormalized mass r vanishes at the quantum critical point, it will not differ much from zero for finite temperatures either. Thus, the propagators of the two modes can be roughly estimated by

$$g_z = \frac{1}{k^2 + \left(\frac{\omega_n}{k}\right)^{2/(z-1)}} \simeq \frac{k^{2/(z-1)}}{(k^z)^{2/(z-1)} + (nT)^{2/(z-1)}} \tag{5.1}$$

where $z = 2, 3$ for the g_2 and g_3, respectively, and n is an integer. The factor of 2π of the Matsubara frequencies is neglected in this qualitative discussion for the sake of clarity.

Now, when the momentum k is larger than $T^{1/z}$ the k^2 term dominates the denominator and the finite spacing of the Matsubara frequencies is irrelevant. This means that one can treat them like a continuous variable which is just the same as in the zero temperature case. One then has a model with the effective dimension $d + z$ and this region in k-space is therefore called the "quantum regime." The system can be treated as it were at zero temperature.

On the other hand, if k is smaller than $T^{1/z}$, the ω_n term dominates the denominator. In this case the spacing between the Matsubara frequencies strongly matters and one can distinguish between the zero and the non-zero Matsubara frequencies

$$g_z(\mathbf{q}, \omega_n) \sim \begin{cases} \frac{1}{k^2} & \text{for } n = 0 \\ \left(\frac{k}{nT}\right)^{2/(z-1)} & \text{for } n \neq 0 \end{cases} \tag{5.2}$$

Since $T > k^z$, the former is much larger than the latter and one may integrate out the non-zero modes perturbative. As a result, one obtains an effective d-dimensional action for the zero mode. Therefore, this procedure is called "*dimensional reduction.*" This low-momentum region is also called the "classical regime," since it is comparable to a classical statistical field theory in d dimensions.

However, the perturbative treatment of the non-zero modes generates an effective mass for the zero mode. A problem arises, if this effective mass becomes of the same order as the non-zero Matsubara term which can be interpreted as the "mass" of the non-zero modes. In the classical regime, one can estimate the lower bound of this mass, by using $k < T^{1/z}$, to $(T/k)^{2/(z-1)} > T^{2/z}$. Therefore, if the effective mass of the zero Matsubara mode is of the order of $T^{2/z}$, the zero mode has no longer an exceptional position, and the perturbative treatment of all other modes would not be justified.

The idea of RG-improved perturbation theory is the following. Since in the high-momentum regime the theory is nearly alike the one at zero temperature, one can

Fig. 5.1:

The different regimes in momentum space for both the g_2 and the g_2 modes. The most interesting region is the striped one, where g_2 is in its quantum regime with dimension $d + z = 4$ and g_2 is in its classical regime with dimension $d = 2$.
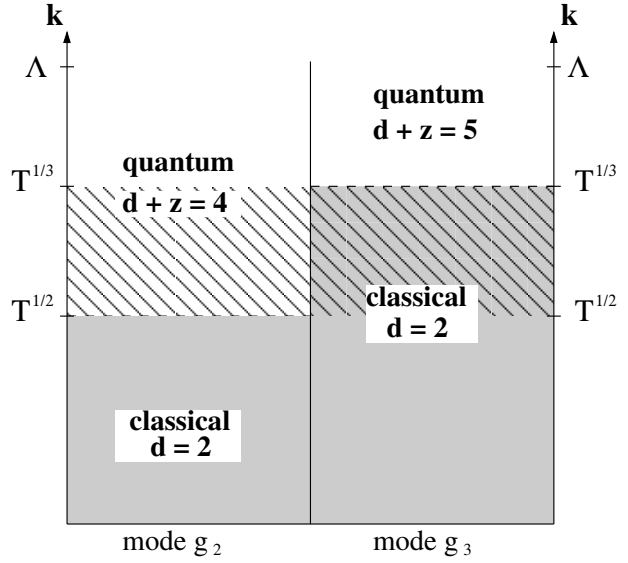
use the previously derived RG flow. The RG flow has to be stopped as soon as one enters the classical regime, i.e. at $b = \Lambda T^{-1/z}$. From the resulting low-momentum theory an effective action for the zero mode is obtained by integrating out all non-zero modes. The effective action is still quartic in the zero mode field but has a non-vanishing mass. Because of this, one can now use plain perturbation theory to solve the problem and get an expression for the zero mode effective mass R. As mentioned before, this mass R has to be compared to $T^{2/z}$ to justify the procedure from hindsight.

A much broader discussion of the idea of dimensional reduction and its application is given in, e.g. , [21].

5.1.2 Dimensional Reduction

If one wants to apply the method described above to the Pomeranchuk theory, the different dynamics raise a problem. Due to the existence of two dynamical exponents there are not two but three different regimes as shown in Fig. 5.1. For momenta larger than $T^{1/3}$ both modes are in the quantum regime, and for momenta smaller than $T^{1/2}$ both modes are in the classical regime. However, for momenta $T^{1/3} < k < T^{1/3}$, the g_2 mode is still in its quantum regime, while the g_3 mode is already in its classical regime. This fact has to be taken into account when generalizing the scheme of dimensional reduction, explained in the previous section. One should be aware that it is this crossover regime which may cause severe problems.

The calculation involves five steps:

- First of all, the bare coupling constants (r_0 and u_0) get renormalized according to the full zero temperature RG equations yielding the renormalized bare coupling constants r and u

37

- In the second step, these renormalized bare coupling constants flow according to the full RG flow until the classical regime of the g_3 mode is reached, i.e. $b = \Lambda T^{-1/3}$. The remaining theory now has a momentum cutoff $T^{1/3}$.

- Then, the coupling constants flow further according to the RG flow which takes *only* the g_2 mode into account. The flow is stopped when the classical regime of the g_2 mode is reached, i.e. $b = \Lambda T^{-1/2}$ yielding r^{R} and u^{R}. Thereby one gets two different cuttoffs: while the g_2 mode has a cutoff $T^{1/2}$ the g_3 mode still has a cutoff $T^{1/3}$.

- Now both modes are in their classical regime and one can integrate out the non-zero Matsubara modes. This gives rise to the effective coupling constants R' and U' for the zero Matsubara mode.

- Finally, one has to get rid of the twofold cutoff structure. Therefore one has to integrate out the intermediate momentum region $T^{1/3} > k > T^{1/2}$ and obtains the effective mass of the zero Matsubara modes

Starting with the outlined procedure, the renormalized bare coupling constants, r and u, flow according to the full zero temperature RG equations until $b = \Lambda T^{-1/3}$. In the intermediate regime one let them flow further according to the zero temperature RG equations, coming from the g_2 mode only. Since in this particular case the asymptotic flow of the full system depends only on the g_2 mode anyway, the flow of the parameters is not different from the second step. The only subtle difference is the already mentioned twofold cutoff structure. This can be denoted by multiplying the inverse propagator g_z^{-1} by a factor of $\Theta(T^{1/z} - k)$. The coupling constants, obtained by this flowing procedure are denoted by r^{R} and u^{R}.

Above the quantum critical point the renormalized bare mass vanishes, $r = 0$, and thereby it is also $r^R = 0$, while for the vertex one obtains

$$u^R = \frac{c}{\ln(e^{c/u}\Lambda \, T^{-1/2})} \tag{5.3}$$

Now one derives an effective action for the zero mode alone. This is done by perturbative calculating the integral

$$e^{S_{\mathrm{eff}}[n_0]} = \int_{n \neq 0} \mathcal{D}\phi_n \, e^{S[n]} \tag{5.4}$$

where $S[n]$ is the action in the classical regime with the renormalized coupling constants.

Although, the RG flow is stopped at the momentum cuttoff $k = T^{1/2}$, the renormalization of the bare mass involves the complete RG flow for arbitrary large values of b. The renormalized bare mass r, therefore, also captures the zero temperature correction for smaller values of k. To avoid double-counting of the corrections from

the low-momentum regime, one has to subtract the zero temperature contribution again.

$$R' = r^{\text{R}} + \frac{4(N+2)K_2}{4!} u^{\text{R}} \left\{ \frac{1}{2\beta} \sum_{\omega_n \neq 0} \left(\int_0^{T^{1/2}} \mathrm{d}q \; q g_2 + \int_0^{T^{1/3}} \mathrm{d}q \; q g_3 \right) \right.$$
$$\left. - \left[\frac{1}{2\beta} \sum_{\omega_n \neq 0} \left(\int_0^{T^{1/2}} \mathrm{d}q \; q g_2 + \int_0^{T^{1/3}} \mathrm{d}q \; q g_3 \right) \right]_{T=0} \right\} \quad (5.5)$$

Note the difference of the upper boundaries of the momentum integrals, due to the Heaviside theta functions $\Theta(T^{1/z} - k)$.

The calculation of this expression, explicitly presented in App. C, yields with $r^{\text{R}} = 0$

$$R' = \frac{(N+2)K_2}{12} u^{\text{R}} \left\{ \int_0^{T^{1/2}} \mathrm{d}q \frac{T}{q} \left[\frac{q^2}{2T} \left(\coth\left(\frac{q^2}{2T} \right) - 1 \right) - 1 \right] \right.$$
$$\left. - \int_0^{T^{1/3}} \mathrm{d}q \frac{T}{q} \left[1 - \frac{q^3}{T} \int \frac{\mathrm{d}\omega}{\pi} \frac{\gamma\omega \left(\coth\left(\frac{\omega}{2T} \right) - 1 \right)}{q^6 + \gamma^2\omega^2} \right] \right\} \quad (5.6)$$

which can be evaluated as

$$R' = u^{\text{R}} T \kappa \sim \frac{T}{\ln(c_1/T)} \quad (5.7)$$

where κ is a constant, and $c_1 = e^{2c/u}\Lambda^2$.

For the vertex correction things can be simplified. While at the quantum critical point the renormalized mass vanishes, the renormalized vertex is finite. An expansion of the effective coupling of the zero Matsubara modes, U, in the temperature therefore yields a finite zeroth order term, $U \approx u^R + c(u^{\text{R}})^2 T$. Thus, higher order terms, as generated by integrating out the non-zero Matsubara modes, are suppressed by a factor of T. Hence, in a reasonable approximation, one may take into account the zero-temperature renormalization only

$$U \approx u^{\text{R}} = \frac{2c}{\ln(c_1/T)} \quad (5.8)$$

Now an effective quartic action for the zero mode is achieved, but it still contains a twofold cutoff structure. Depending on which mode one considers the cutoff is either $T^{1/3}$ or $T^{1/2}$. In order to get an effective action with only one cutoff, one has to integrate out the momenta $T^{1/3} > k > T^{1/2}$ sector, which yields another correction to the effective mass

$$R = R' + \frac{(N+2)K_2}{12} U \int_{T^{1/2}}^{T^{1/3}} \mathrm{d}q \frac{g_3}{2}$$
$$= R' + \frac{(N+2)K_2}{12} u^{\text{R}} \frac{T}{2} \ln\left(\frac{R' + T^{1/3}}{R' + T^{1/2}} \right) \quad (5.9)$$

39

Substituting the expression for R' and u^R one finally gets the effective mass for the zero modes

$$R = \text{const} \cdot T + \text{const} \cdot \frac{T}{\ln(1/T)} \tag{5.10}$$

To leading order this mass grows linearly with temperature. Note, that the linear temperature dependence originates from the overlap regime only. To sum up, the effective action for the zero mode reads

$$S[n_0] = \frac{1}{2}\frac{T}{L^d}\sum_{\mathbf{k}} n^{\text{i}}_{0,\mathbf{k}}(R + k^2)n^{\text{i}}_{0,\mathbf{k}} + \frac{U}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}} n^{\text{i}}_{0,\mathbf{k_1}} n^{\text{i}}_{0,\mathbf{k_2}} n^{\text{j}}_{0,\mathbf{k_3}} n^{\text{j}}_{0,-\mathbf{k_1}-\mathbf{k_2}-\mathbf{k_3}} \tag{5.11}$$

Further corrections of the physical mass from the quartic term are additive and therefore only increase the leading-order behavior.

In the last step, one has to verify the consistency of this approach by comparing the effective mass of the zero mode to the masses of the non-zero modes. The mass of the non-zero modes is of the order of $T^{2/z}$. Thus, the effective mass of the zero modes is much smaller than the mass of the non-zero modes with $z = 3$. However, the non-zero modes with $z = 2$ have a mass comparable to the effective zero-mode mass. As pointed out before, this conflicts with the special treatment of the zero mode, and thus, the concept of dimensional reduction breaks down. The approach is inconsistent. It has to be stressed that up to Eq. (5.7) the effective mass, R', is logarithmically suppressed. The consistency condition is violated only due to the contribution of the overlap regime.

5.2 Renormalization Group Approach

The calculation of the previous section has shown, that one cannot approach the problem with perturbation theory. Therefore, one once again uses renormalization group methods to obtain the finite temperature correlation length.

In this section, the RG equations at small but finite temperatures are derived according to the scheme developed in Chap. 4. From the RG flow of the mass one can derive the correlation length within the quantum critical region. In order to simplify calculations and to get a feeling for the result, one may first consider what can be expect from the outset.

The region of the phase diagram in which one is interested is the regime right above the quantum critical point. Here, the renormalized r which covers the RG flow of r_0 at zero temperature vanishes. In the vicinity of the quantum critical point it differs only slightly from zero which implies that temperature dependend corrections are important. In contrast, the zero temperature coupling constant u has a

finite value at the quantum critical point, and an expansion of the RG flow in temperature yields only a small correction. Therefore, one can neglect the temperature dependence of the vertex and approximate it by its zero temperature RG flow.

The second consideration is a more qualitative one. As pointed out before, the momentum space can be separated into the three regimes depicted in Fig. 5.1. In the large-momentum region, both modes are of quantum nature, having almost continuous Matsubara frequencies and an effective dimension $d + z$. In the small-momentum region, the whole theory is classical, i.e. comparable to a classical d-dimensional field theory.

In the interesting regime, where the g_3 mode is already classical, while the g_2 mode is still in the quantum regime, the g_3 mode is at the lower critical dimension $d = 2$. On the other hand, the g_2 mode is at the upper critical dimension $d + z = 4$. Since the flow of u is determined by the g_2 mode only while the finite temperature correction of the mass will be governed by the g_3 mode, the most divergent contributions come from this overlap regime. This means that the theory is dominated by the momentum scale between $T^{1/3}$ and $T^{1/2}$. The strong influence of this momentum regime can be also seen from Eq. 5.9 and Eq. 5.10. It is the overlap regime that leads to the failure of dimensional reduction.

5.2.1 Calculation of the β Function at Finite Temperature

As explained before, at finite temperature one only has to calculate the RG equation for the mass r. The vertex u is in leading order governed by its zero temperature RG flow

$$u(b) = \frac{c}{\ln\left(be^{c/u}\right)} \qquad \text{with } c = \frac{16}{3K_2} \tag{5.12}$$

The one-loop correction to the low momentum action is the same as in the zero temperature case

$$\langle S_{\text{int}}[n_{\text{h}}, n_{\text{l}}]\rangle_{\text{h}} = 2\frac{u_0}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{k_1, k_2, k_3} \langle n_{k_1}^{\text{i}} n_{k_2}^{\text{i}}\rangle_{\text{h}} n_{k_3}^{\text{j}} n_{-k_1-k_2-k_3}^{\text{j}}$$

$$+ 4\frac{u}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{k_1, k_2, k_3} n_{k_1}^{\text{i}} \left\langle n_{k_2}^{\text{i}} n_{k_3}^{\text{j}}\right\rangle_{\text{h}} n_{-k_1-k_2-k_3}^{\text{j}} \tag{5.13}$$

but this time the sum over $k = (\mathbf{k}, \omega_n)$ extends over the discrete bosonic Matsubara frequencies $\omega_n = 2\pi nT$ instead of continuous ones. This can be rewritten as shown in Chap. 4 to yield

$$\langle S_{\text{int}}[n_{\text{h}}, n_{\text{l}}]\rangle_{\text{h}} = \frac{2(N+2)u_0}{4!}\left(\frac{T}{L^d}\right)^2 \sum_{k,k'} \frac{g_2(k') + g_3(k')}{2} n_k^i n_{-k}^i \tag{5.14}$$

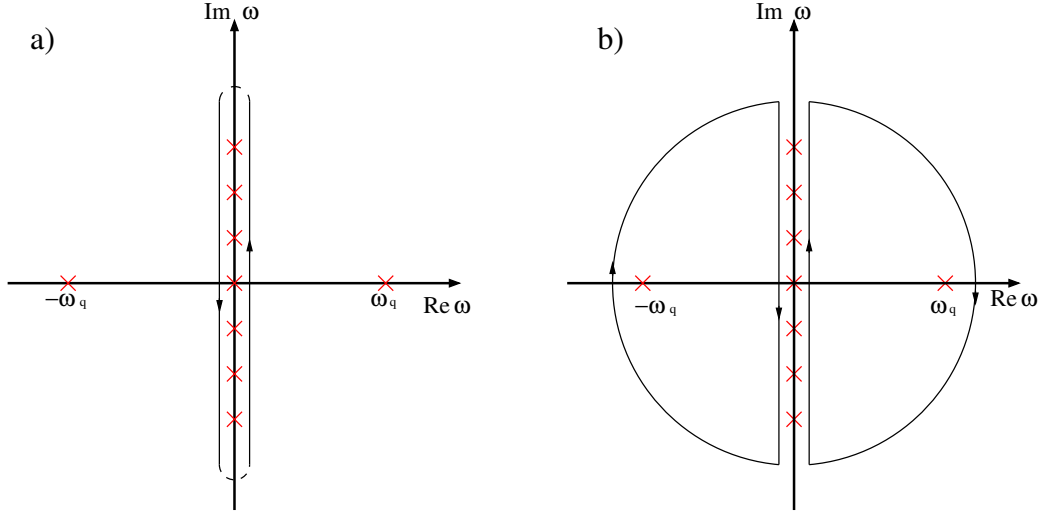In the following, only the main ideas are presented while the actual calculation is performed in detail in App. D.

Fig. 5.2: Illustration of the integration contours for the g_2 mode. On the left side the contour C is depicted. This contour is deformed into the contour C' depicted on the right side.

To calculate the sum over the g_2 contribution, one applies the residue theorem to convert the sum over Matsubara frequencies into an integral

$$\frac{T}{L^d} \sum_{\mathbf{q},\omega_n} g_2(\mathbf{q},\omega_n) = \int \frac{\mathrm{d}^2q}{(2\pi)^2} T \sum_{\omega_n} \frac{q^2}{q^2(r+q^2)+\omega_n^2}$$

$$= \int \frac{\mathrm{d}^2q}{(2\pi)^2} \frac{1}{2} \int_C \frac{\mathrm{d}\omega}{2\pi i} \coth\left(\frac{\omega}{2T}\right) \frac{q^2}{q^2(r+q^2)-\omega^2} \tag{5.15}$$

Here, the integration contour C is depicted in Fig. 5.2 (a) Deforming this contour into the contour C' as shown in Fig. 5.2 (b), one only has to sum up the residues at the poles $\pm\omega_q = \pm q\sqrt{r+q^2}$ which leads to

$$\frac{T}{L^d} \sum_{\mathbf{q},\omega_n} g_2(\mathbf{q},\omega_n) = \frac{1}{2} \int \frac{\mathrm{d}^2q}{(2\pi)^2} \coth\left(\frac{\omega_q}{2T}\right) \frac{q^2}{\omega_q} \tag{5.16}$$

On the other hand, for the g_3 mode one first gets rid of the absolute value of ω by summing twice over the positive Matsubara frequencies and adding the ω_0 term

$$\frac{T}{L^d} \sum_{\mathbf{q},\omega_n} g_3(\mathbf{q},\omega_n) = T \int \frac{\mathrm{d}^2q}{(2\pi)^2} \left[\frac{1}{r+q^2} + 2 \sum_{\omega_n>0} \frac{q}{q(r+q^2)+\gamma_0\omega_n} \right] \tag{5.17}$$

Similar as to the g_2 mode one applies the residue theorem to rewrite the Matsubara frequency sum into an integral over the contour S shown in Fig. 5.3 (a). This contour is then deformed into the contour S' depicted in Fig. 5.3 (b) on the right. Since the large semi-circle vanishes for a large frequency cutoff Γ, and the infinitesimal semi-circle with radius $\epsilon \to 0$ cancels against the zeroth Matsubara mode contribution, one is left with the relation

$$\frac{T}{L^d} \sum_{\mathbf{q},\omega_n} g_3(\mathbf{q},\omega_n) = \int \frac{\mathrm{d}^2q}{(2\pi)^2} \, \mathcal{P}\int_{-\Gamma}^{\Gamma} \frac{\mathrm{d}\omega}{2\pi i} \coth\left(\frac{\omega}{2T}\right) \frac{q}{q(r+q^2)-i\gamma_0\omega} \tag{5.18}$$
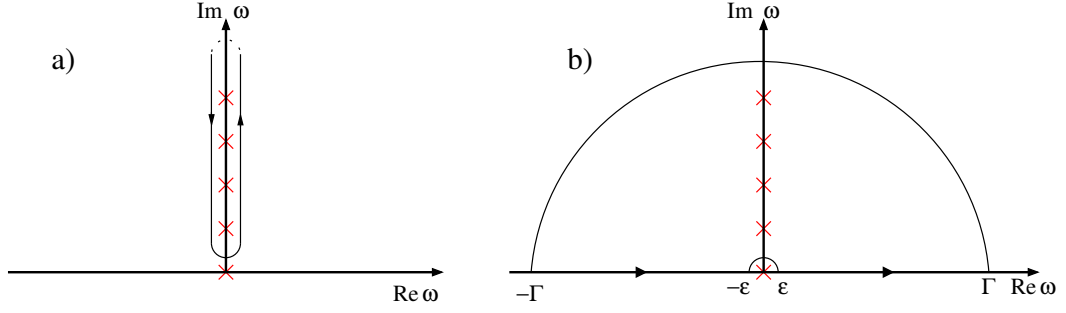
42

Fig. 5.3: Illustration of the integration contours for the g_3 mode. On the left side the contour S is depicted. This contour is deformed into the contour S' depicted on the right side.

Here, $\mathcal{P}$ denotes the principal value integral since $\omega = \omega_0 = 0$ is of the domain of integration.

In order to obtain the β function, one has to proceed in the same way as for the zero temperature case. After restricting the integrals to the high momentum/frequency shell, one takes the derivative with respect to $\ln b$ at $b = 1$. Thus, the contributions for the g_2 and g_3 mode read as

$$f_2(r, \gamma_0, T) = K_2 \Lambda^2 \frac{\Lambda^2}{2\omega_\Lambda} \coth\left(\frac{\omega_\Lambda}{2T}\right) \tag{5.19}$$

$$f_3(r, \gamma_0, T) = K_2 \Lambda^3 \int_0^\Gamma \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{\gamma_0 \omega}{\Lambda^{-2}\omega_\Lambda^4 + \gamma_0^2 \omega^2}$$

$$+ \frac{z\Gamma^2}{\pi} \int_0^\Lambda dq \coth\left(\frac{\Gamma}{2T}\right) \frac{q^2 \gamma_0}{q^{-2}\omega_q^4 + \gamma_0^2 \Gamma^2} \tag{5.20}$$

where the notation $\omega_\Lambda = \Lambda\sqrt{r + \Lambda^2}$ was used. Instead of writing down the RG equation for r, one can absorb its engineering dimension by defining $R = rb^{-2}$. Note that this R must not be confused with the the one defined in the previous section. The β function for R takes the form

$$\frac{\partial R}{\partial \ln b} = \frac{2(N+2)}{4!} u(b)b^{-2} \left[f_2(Rb^2, \gamma(b), T(b)) + f_3(Rb^2, \gamma(b), T(b))\right] \tag{5.21}$$

The temperature which is proportional to ω_n has to flow according to $T(b) = Tb^z$, while γ flows like in the zero temperature case according to $\gamma(b) = \gamma_0 b$

5.2.2 Correlation Length Above the Quantum Critical Point

The correlation length ξ is connected to the mass as $\xi^{-2} \sim R(b_\star)$ where $b_\star$ is the point at which the RG flow stops. To extract $R(b)$, the RG equation can be integrated over $d(\ln b) = \frac{db}{b}$ yielding

$$R(b_\star) - R(1) = \frac{2(N+2)K_2}{4!} \int_1^{b_\star} \frac{db'}{b'} u(b')b'^{-2} \left[f_2(Rb'^2, \gamma(b'), T(b'))\right.$$

$$\left. + f_3(Rb'^2, \gamma(b'), T(b'))\right] \tag{5.22}$$

43

At the quantum critical point the zero temperature renormalization of the bare mass $R(1)$ vanishes so that

$$R(b_\star) = \underbrace{R(1) + \int_1^{b_\star} \frac{db'}{b'} F(b', R, T)\big|_{T=0}}_{=0} + \int_1^b \frac{db'}{b'} \left[F(b', R, T) - F(b', R, 0) \right] \quad (5.23)$$

where $F(b', R, T)$ denotes the integration kernel.

Substituting $b = \Lambda/k$, rescaling $\omega \to \omega k^2/\Lambda^2$ and $q \to q\Lambda/k$ with the associated cutoffs Γ' and k, and inserting the zero temperature result for $u(b)$ one finally gets $(z = 2)$

$$R(b_\star) = \frac{8}{9} \int_{\Lambda/b_\star}^{\Lambda} \frac{dk}{\ln(e^{c/u}\Lambda/k)} \left[\frac{k^2}{\sqrt{R(b) + k^2}} \left(\coth\left(\frac{k\sqrt{R(b) + k^2}}{2T} \right) - 1 \right) \right.$$
$$+ 2k^2 \int_0^{\Gamma'} \frac{d\omega}{\pi} \frac{\gamma_0 \omega \left(\coth\left(\frac{\omega}{2T} \right) - 1 \right)}{k^2(R(b) + k^2)^2 + \gamma_0^2\omega^2}$$
$$\left. + 2\Gamma' \frac{2}{\pi} \int_0^k dq \frac{q^2\gamma_0 \left(\coth\left(\frac{\Gamma'}{2T} \right) - 1 \right)}{q^2(R(b) + q^2)^2 + \gamma_0^2\Gamma'^2} \right]_{b=\Lambda/k} \quad (5.24)$$

The RG flow stops at $b_\star = \Lambda/\sqrt{R}$. Thus, the equation becomes self-consistent.

A detailed analysis of the individual terms (see App. D) shows that the leading order contribution originates only from the zero Matsubara mode ω_0, and the k-integration may be restricted to the interval $[R^{1/2}, T^{1/3}]$. In this regime one can approximate $R(\Lambda/k) + k^2 \approx k^2$ and finally ends up with the two terms

$$R = \frac{8}{9} \left\{ \int_{R^{1/2}}^{T^{1/2}} \frac{dk}{k} \frac{2T}{\ln(c_1/k)} + \int_{R^{1/2}}^{T^{1/3}} \frac{dk}{k} \frac{2T}{\ln(c_1/k)} \right\} \quad (5.25)$$

where $c_1 = \Lambda e^{c/u}$. Carrying out the integrations, a self-consistent equation determining the correlation length at finite temperatures arises

$$\xi^{-2} = R = -\frac{16}{9} T \left[\ln\left(\frac{\ln\left(c_1/T^{1/2}\right)}{\ln\left(c_1/R^{1/2}\right)} \right) + \ln\left(\frac{\ln\left(c_1/T^{1/3}\right)}{\ln\left(c_1/R^{1/2}\right)} \right) \right] \quad (5.26)$$

Approximating $R = \bar{c}\, T$ on the right hand side, one can fairly neglect the first logarithm which leads to a correction of order $1/\ln T$. One thus gets

$$\xi^{-2} = \frac{16}{9} T \left[\ln\left(\frac{\ln\left(c_1/T^{1/3}\right)}{\ln\left(c_1/(\bar{c}T)^{1/2}\right)} \right) \right] \approx \frac{16}{9} T \left[\ln\frac{3}{2} + \mathcal{O}\left(\frac{1}{\ln(1/T)} \right) \right] \quad (5.27)$$

Interestingly this result shows that the effective mass grows to leading order linearly with temperature without *any* logarithmic corrections. Even more astonishing is the fact that the leading-order behavior has a prefactor of $\frac{16}{9}\ln(3/2)$ which is completely independent of the interaction strength u.
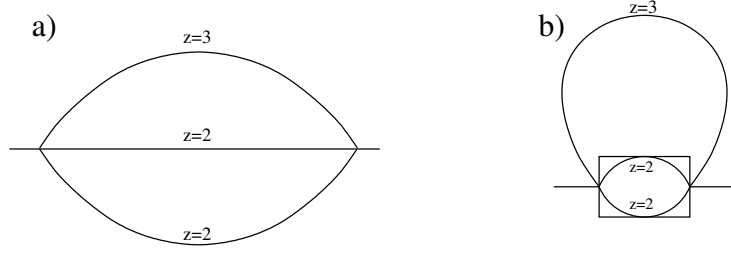
Fig. 5.4: The most important second order diagram. On the right hand the same diagram is displayed but the $z = 2$ modes are gathered into an effective vertex, denoted by a box.

5.3 Calculation of a Second Order Diagram

In the previous section, the leading logarithmic divergencies were resummed within the RG method. This was done by a scheme where all frequencies were rescaled according to $z = 2$. In principle, the coupling constant u for the low frequency/momentum sector is a function of three Matsubara frequencies and momenta. However, within this approach one assumes that the flow of u can be captured by one parameter, b, only. The validity of this assumption for the applied scheme has to be checked.

In order to verify the results, one may calculate the most important two-loop diagram and obtain the leading-order temperature dependence. Due to the number of free summations one identifies the diagram shown in Fig. 5.4 (a) as the most important one. It can be interpreted as a normal Hartree diagram with a one-loop renormalized vertex, as shown in Fig. 5.4 (b). As mentioned above, the finite temperature mass correction is governed by the g_3 mode while the most divergent vertex correction comes from the g_2 mode. One thus expects the leading-order contribution to be the diagram where two of the propagators are ballistic and only one propagator is damped, which will be called the 223-contribution from now on.

The general expression for the displayed diagram with no external momentum reads

$$\Delta R_{ij} = -4^3 \left(\frac{u}{4!}\right)^2 \left(\frac{T}{L^d}\right)^3 \sum_{k,q,p} \left[\frac{1}{2} g_{ij}(k) g_{mn}(q) g_{nm}(p) + g_{im}(k) g_{mn}(q) g_{nj}(p)\right] \delta_{k+q+p} \tag{5.28}$$

where it was used that the matrix g is symmetric, and the summation over indices m and n is implicit. After angular integration one, of course, expects ΔR to be diagonal, i.e. $\Delta R_{i,j} = \Delta R \, \delta_{ij}$.

Rewriting $g(k) = g_2(k) U_2(\hat{\mathbf{k}}) + g_3(k) U_3(\hat{\mathbf{k}})$ with the matrices U_z defined in Chap. 3, one can separate all 223-contributions and gets from Eq. 5.28

$$\Delta R_{223} = \left(\frac{T}{L^d}\right)^3 \sum_{k,q,p} g_3(k) g_2(q) g_2(p) P(\mathbf{k}, \mathbf{q}, \mathbf{p}) \delta_{k+q+p} \tag{5.29}$$

45

Here, the matrix $P(\mathbf{k}, \mathbf{q}, \mathbf{p})$ was defined as

$$P(\mathbf{k}, \mathbf{q}, \mathbf{p}) = -4^3 \left(\frac{u}{4!} \right)^2 \left[\frac{1}{2} U_3(\hat{\mathbf{k}}) \, \text{Tr} \left(U_2(\hat{\mathbf{q}}) U_2(\hat{\mathbf{p}}) \right) + U_2(\hat{\mathbf{q}}) \, \text{Tr} \left(U_3(\hat{\mathbf{k}}) U_2(\hat{\mathbf{p}}) \right) \right.$$
$$\left. + U_3(\hat{\mathbf{k}}) U_2(\hat{\mathbf{q}}) U_2(\hat{\mathbf{p}}) + U_2(\hat{\mathbf{q}}) U_3(\hat{\mathbf{k}}) U_2(\hat{\mathbf{p}}) + U_2(\hat{\mathbf{p}}) U_2(\hat{\mathbf{q}}) U_2(\hat{\mathbf{k}}) \right] \tag{5.30}$$

Defining $\Pi(\mathbf{q}, \mathbf{p}, \omega_n) = T^2 \sum_{\Omega_n, \Omega_m} g_2(\mathbf{q}, \Omega_n) g_2(\mathbf{p}, \Omega_m) \delta(\Omega_n + \Omega_m + \omega_n)$ and performing the analytical continuation ($i\omega_n \to w + i0$) one gets

$$\Pi(\mathbf{q}, \mathbf{p}, \omega) = \frac{q^2 p^2}{4 \omega_q \omega_p} \left[\omega_p \coth \frac{\omega_q}{2T} \left(\frac{1}{\omega_p^2 - (\omega_q + \omega + i0)^2} + \frac{1}{\omega_p^2 - (\omega_q - \omega - i0)^2} \right) \right.$$
$$\left. + \omega_q \coth \frac{\omega_p}{2T} \left(\omega_q \leftrightarrow \omega_p \right) \right] \tag{5.31}$$

where, as before, $\omega_q = q\sqrt{r + q^2}$.

The remaining Matsubara summation can be calculated using the residue theorem

$$T \sum_{\omega_n} g_3(\omega_n) \Pi(\omega_n) = \int \frac{\mathrm{d}\omega}{2\pi} \coth \left(\frac{\omega}{2T} \right) \left[\text{Im}\, \tilde{g}_3(\omega_n) \, \text{Re}\, \Pi(\omega_n) \right.$$
$$\left. + \text{Re}\, \tilde{g}_3(\omega_n) \, \text{Im}\, \Pi(\omega_n) \right]_{i\omega_n \to w + i0} \tag{5.32}$$

Here, $\tilde{g}_3(-i\omega) = (r + q^2 - i\gamma_0 \omega / q)$ denotes the analytical continuation in the upper complex plain which differs from the lower complex plain by a minus sign in front of the frequency ω. For the sake of brevity, $\tilde{g}_3$ is further on also called g_3.

With these definitions one can rewrite Eq. 5.29 as

$$\Delta R_{223} = L^{-3d} \sum_{\mathbf{k}, \mathbf{q}, \mathbf{p}} P(\mathbf{k}, \mathbf{q}, \mathbf{p}) \delta_{\mathbf{k} + \mathbf{q} + \mathbf{p}} \int \frac{\mathrm{d}\omega}{2\pi} \coth \left(\frac{\omega}{2T} \right)$$
$$\times \left[\text{Im}\, g_3(\mathbf{k}, \omega_n) \, \text{Re}\, \Pi(\mathbf{q}, \mathbf{p}, \omega_n) + \text{Re}\, g_3(\mathbf{k}, \omega_n) \, \text{Im}\, \Pi(\mathbf{q}, \mathbf{p}, \omega_n) \right]_{i\omega_n \to w + i0} \tag{5.33}$$

5.3.1 Recovering the RG Result

Starting with the first term of the integral in Eq. 5.33, one has to calculate

$$\Delta R_{223} = L^{-3d} \sum_{\mathbf{k}, \mathbf{q}, \mathbf{p}} \int \frac{\mathrm{d}\omega}{2\pi} \coth \left(\frac{\omega}{2T} \right) \left[\text{Im}\, g_3(\mathbf{k}, \omega_n) \, \text{Re}\, \Pi(\mathbf{q}, \mathbf{p}, \omega_n) \right]_{i\omega_n \to w + i0} P(\mathbf{k}, \mathbf{q}, \mathbf{p}) \delta_{\mathbf{k} + \mathbf{q} + \mathbf{p}} \tag{5.34}$$

Concerning the integration over $\mathbf{q}$ and $\mathbf{p}$ one gets

$$\delta u(\mathbf{k}, r, T, \omega) = \sum_{\mathbf{q}, \mathbf{p}} \text{Re}\, \Pi(\mathbf{q}, \mathbf{p}, \omega) P(\mathbf{k}, \mathbf{q}, \mathbf{p}) \delta_{\mathbf{k} + \mathbf{q} + \mathbf{p}} = \int_0^\Lambda \frac{\mathrm{d}q}{q} F \left(\frac{q}{k}, \frac{q}{r^{1/2}}, \frac{q}{T^{1/2}}, \frac{q}{\omega^{1/2}}, x \right) \tag{5.35}$$

where $x = \phi_{\mathbf{k}} - \phi_{\mathbf{q}}$ is the angle between $\mathbf{k}$ and $\mathbf{q}$. If all four ratios are larger than one, the function F resembles the k-independent result $c \int d^d q \, d\omega \, g_2(q, \omega) g_2(q, \omega)$ calculated in Sect. 5.2. If one of these ratios is smaller than one, one gets a less divergent behavior which is not important for the leading order calculation. Thus, one obtains

$$\delta u(\mathbf{k}, r, T, \omega) = -3 \frac{u^2 K_2}{8\pi} \int_c^\Lambda dq \int d\omega' \, g_2(q, \omega') g_2(q, \omega')$$
$$= -3 \frac{u^2 K_2}{16\pi} \ln \frac{\Lambda}{c} \tag{5.36}$$

where the lower momentum cutoff is $c = \max[k, r^{1/2}, T^{1/2}, \omega^{1/2}]$.

Now one substitutes this result into the leading order contribution of the remaining $\mathbf{q}$ and ω integral.

$$\Delta R_{223} \approx \frac{K_2}{3} \int_0^\Lambda dk \, k \int_0^\Gamma \frac{d\omega}{\pi} \coth \frac{\omega}{2T} \delta u(k, r, T, \omega) k \frac{\gamma_0 \omega}{k^{-2} \omega_k^4 + \gamma_0^2 \omega^2} \tag{5.37}$$

The last fraction is peaked at $\gamma_0 \omega \sim k^{-1} \omega_k^2$. Therefore, the frequency is to leading order pinned to $\omega \sim k^3$ which implies that δu does not depend on ω. Subtracting the zero temperature contribution, one can approximate

$$\coth \left(\frac{k^{-1} \omega_k^2}{2\gamma_0 T} \right) - 1 \approx \begin{cases} 2\gamma_0 T \frac{k}{\omega_k^2} & \text{for } k \leq (2\gamma_0 T)^{1/3} \\ 0 & \text{for } k \geq (2\gamma_0 T)^{1/3} \end{cases} \tag{5.38}$$

One finally arrives in leading order at

$$\Delta R_{223} \approx \frac{K_2}{3} T \int_0^{(2\gamma_0 T)^{1/3}} dk \, \delta u(k, r, T) \frac{k^3}{\omega_k^2} \tag{5.39}$$

Note that the momentum cutoff $\sim T^{1/3}$ originates from the g_3 mode. For $k^2 < T, r$ the vertex δu is k-independent while in the region $k^2 > T, r$ the vertex is essentially given by $\delta u \sim \ln k$. Therefore, the leading order behavior comes from this second interval. Assuming that $r \sim T < k^2$, one can approximate $\omega_k \approx k^2$ and arrives at

$$\Delta R_{223} \approx - \left(\frac{u K_2}{4} \right)^2 \int_{r^{1/2}}^{(2\gamma_0 T)^{1/3}} \frac{dk}{k} \ln \frac{\Lambda}{k}$$
$$\approx - \left(\frac{u K_2}{4} \right)^2 \frac{T}{2} \left[\ln^2 \left(\frac{\Lambda}{r^{1/2}} \right) - \ln^2 \left(\frac{\Lambda}{(2\gamma_0 T)^{1/3}} \right) \right] \tag{5.40}$$

Expansion of the RG result of R, cf. Eq. (5.27), up to second order in u_0 (which is contained in c_1) yields the same result. Therefore, the applied RG scheme seems to sum up the leading divergences in the correct manner. However, to prove this statement, the second term in Eq. (5.32) has to be calculated, although from naive scaling arguments one may infer that it should be of the order of T^2.

5.3.2 Interplay of the Two Modes

The second term of Eq. (5.33) reads

$$\Delta R_{223} = L^{-3d} \sum_{\mathbf{k},\mathbf{q},\mathbf{p}} \int \frac{\mathrm{d}\omega}{2\pi} \coth\left(\frac{\omega}{2T}\right) \left[\mathrm{Re}\, g_3(\mathbf{k},\omega_n) \,\mathrm{Im}\,\Pi(\mathbf{q},\mathbf{p},\omega_n) \right]_{i\omega_n \to w+i0} P(\mathbf{k},\mathbf{q},\mathbf{p}) \delta_{\mathbf{k}+\mathbf{q}+\mathbf{p}}$$

$$(5.41)$$

Taking the imaginary part of Π in Eq. (5.31) yields Dirac delta functions for frequencies

$$\mathrm{Im}\,\Pi(\mathbf{q},\mathbf{p},\omega) = \frac{q^2 p^2}{8\omega_q \omega_p} \left[\left(\coth\frac{\omega_q}{2T} + \coth\frac{\omega_p}{2T} \right) \left(\delta(\omega - \omega_p - \omega_q) - \delta(\omega + \omega_p + \omega_q) \right) \right.$$
$$\left. + \left(\coth\frac{\omega_q}{2T} - \coth\frac{\omega_p}{2T} \right) \left(\delta(\omega - \omega_p + \omega_q) - \delta(\omega + \omega_p - \omega_q) \right) \right]$$

$$(5.42)$$

The two Dirac delta functions in the first line correspond to processes where two quasiparticles with energies ω_q and ω_p are absorbed or emitted. These processes also take place at zero temperature where the prefactor in the parentheses becomes a factor of two. In contrast, the processes of the second line cannot take place at zero temperature due to the negative sign in the prefactor. These processes are simultaneous absorption and emission of quasiparticles with energies ω_q and ω_p. Nevertheless, at small but finite temperatures the latter process is the dominating one.

Performing the integration (5.32) and subtracting the zero temperature contribution leads to the relation

$$\Delta R_{223} - \Delta R_{223} \Big|_{T=0} = \int_0^\Lambda \frac{\mathrm{d}^d k \ \mathrm{d}^d q \ \mathrm{d}^d p}{(2\pi)^2 (2\pi)^2 (2\pi)^2} \left(\coth\frac{\omega_q}{2T} \coth\frac{\omega_p}{2T} - 1 \right) \frac{q^2 p^2 \omega_k^2}{8\omega_q \omega_p} P(\mathbf{k},\mathbf{q},\mathbf{p})$$
$$\times \left(\frac{1}{k^{-2}\omega_k^4 + \gamma_0^2(\omega_q + \omega_p)^2} + \frac{1}{k^{-2}\omega_k^4 + \gamma_0^2(\omega_q - \omega_p)^2} \right) \delta_{\mathbf{k}+\mathbf{q}+\mathbf{p}}$$

$$(5.43)$$

The first term in the second line correspond to the twofold emission or absorption processes whereas the second fraction resembles the mixed absorption-emission processes.

To obtain the leading order behavior, one notices that for $\omega_q = \omega_p$ the second fraction diverges with a higher power of momentum. Since, this is a whole subspace of the integration regime, one can neglect the first term and approximate the second term by

$$\frac{1}{k^{-2}\omega_k^4 + \gamma_0^2(\omega_q - \omega_p)^2} \approx \frac{\pi k}{\gamma_0 \omega_k^2} \delta(\omega_q - \omega_p)$$

$$(5.44)$$

Performing the integration over $\mathbf{p}$ and anticipating the further angular integrations one obtains

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \delta_{\mathbf{k}+\mathbf{q}+\mathbf{p}} P_{ij}(\mathbf{k}, \mathbf{q}, \mathbf{p}) = -\left(\frac{u}{3!}\right)^2 \left(3 - 4\cos 4x + 2\cos 4(\phi_{-\mathbf{k}-\mathbf{q}} - \phi_{\mathbf{q}})\right) \delta_{ij}$$

$$(5.45)$$

where the angle $x = \phi_{\mathbf{k}} - \phi_{\mathbf{q}}$ was introduced, and the momentum $\mathbf{p}$ has been replaced by $-(\mathbf{k} + \mathbf{q})$. Since P is a multiple of the identity matrix, a matrix notation is no longer necessary. In the following, ΔR_{223} denotes the scalar mass correction.

The Dirac delta function of Eq. (5.44) demands that $|\mathbf{q}|$ and $|\mathbf{p}| = |\mathbf{k} + \mathbf{q}|$ are equal, which implies $\phi_{-\mathbf{k}-\mathbf{q}} - \phi_{\mathbf{q}} = 2x$. Converting the Dirac delta function over energies into one for the absolute value of $\mathbf{k}$ and putting everything together one arrives at

$$\Delta R_{223} - \Delta R_{223}\Big|_{T=0} = -\left(\frac{u}{3!}\right)^2 \frac{\pi}{4\gamma_0} \int_0^\Lambda \mathrm{d}q\mathrm{d}k \int_0^{2\pi} \frac{\mathrm{d}x}{(2\pi)^2} \frac{\mathrm{d}\phi_{\mathbf{q}}}{(2\pi)^2} \left(\coth^2 \frac{\omega_q}{2T} - 1\right) \frac{q^4}{\omega_q^2} \frac{q^2 k}{\mathrm{d}\omega_q/\mathrm{d}q}$$

$$\times \left(3 - 4\cos 4x + 2\cos 8x\right) \delta(k + 2q\cos x) \quad (5.46)$$

Performing the integration over k and $\phi_{\mathbf{q}}$ restricts the x integration to the interval of $[\frac{\pi}{2}, \frac{3\pi}{2}]$, and one obtains

$$\Delta R_{223} - \Delta R_{223}\Big|_{T=0} = \left(\frac{uK_2}{3!}\right)^2 \frac{1}{4\gamma_0} \int_0^\Lambda \mathrm{d}q \int_{3\pi/2}^{\pi/2} \mathrm{d}x \left(\coth^2 \frac{\omega_q}{2T} - 1\right) \frac{q^4}{\omega_q^2} \frac{q^3}{\mathrm{d}\omega_q/\mathrm{d}q}$$

$$\times \cos x \left(3 - 4\cos 4x + 2\cos 8x\right)$$

$$(5.47)$$

Finally, rescaling $q^2 \rightarrow 2T\, q^2$ and introducing $\bar{\omega}_q = q(\frac{r}{2T} + q^2)^{1/2}$, one may approximate $\Lambda T^{-1/2} \rightarrow \infty$ for small temperatures and ends up with

$$\Delta R_{223} - \Delta R_{223}\Big|_{T=0} = -\left(\frac{uK_2}{3!}\right)^2 \frac{2038}{315} \frac{T^{3/2}}{\sqrt{2}\gamma_0} \left(\int_0^\infty \mathrm{d}q \left(\coth^2 \bar{\omega}_q - 1\right) \frac{q^4}{\bar{\omega}_q^2} \frac{q^3}{\mathrm{d}\bar{\omega}_q/\mathrm{d}q}\right)$$

$$(5.48)$$

where the integral is finite and depends on r/T via ω_q. Using the RG result obtained in Sect. 5.2 for the renormalized mass, $r = 16/9 \ln(3/2)\, T$, the energy ω_q is temperature independent, and, thus, the integral gives just a number.

In general, using the RG result for r demands the use of the momentum dependent RG result for u. However, due to the form of the integrand the major integration regime is given by $[r^{1/2}, T^{1/2}]$. This is a parametrically small regime and therefore one must not take the flow of u into account.

The important thing to notice is the temperature dependence $T^{3/2}$ of the correction whereas one initially expected only a quadratic behavior. This term arises from a non-trivial interplay of the two modes and relies crucially on the different

dynamical exponents. If there would be only one dynamical exponent, the approximation in Eq. (5.44) would not be justified. Physically, this means that if one probes the vertex with an energy ω, it responds at finite temperatures with a simultaneous emission and absorption of quasiparticles both with the same energy ω_q. This effect can be seen as some kind of resonance. However, at zero temperature this process can not occur.

For the developed RG scheme the consequences are twofold. On the one hand, the latter derivation shows that there are no equally divergent terms correcting the prefactors in Eq (5.40). Thus, the RG scheme seems to sum up the leading divergences of each order. In this respect, the scheme works well. On the other hand, there are other, subleading corrections, which stem from the interplay of the different dynamical exponents. These corrections cannot be captured within the current scheme, even at arbitrary loop order. This is because the scheme handles the problem as if there were only one dynamical exponent. The remaining one is hiden within a flowing number γ. Hence, the RG scheme may work for some quantities but is not a general tool for all calculations.

5.4 Summary

In this chapter, the model of a fermionic system near a quadrupolar Pomeranchuk instability was investigated at finite temperatures. As it turns out, this model has some peculiar properties.

Most notably, the system is governed by the momentum scale between $T^{1/3}$ and $T^{1/2}$ only. Here, the g_2 mode is characterized by its quantum nature while the g_3 mode is already classical. The interplay between the two modes in this regime is crucial and leads to unexpected results. In particular, it is this overlap regime which inhibits the success of the concept of dimensional reduction. As shown in Sect. 5.1, the approach to derive an effective action for the zero Matsubara modes by integrating out all other modes failed. The self-consistency requirement was not met since the effective mass of the zero Matsubara mode was of the same order as the mass of non-zero Matsubara modes with $z = 2$. This inconsistency originates only from the last step of the calculation, where the overlap regime was integrated out.

Since dimensional reduction failed, the system was once again explored by means of RG techniques. In Sect. 5.2 the previously established RG scheme was applied to the model. As a result the inverse square of the correlation length at finite temperatures was obtained to be to first order linear in temperature. Logarithmic corrections appeared only for subleading terms. Furthermore, the leading order term was universal, i.e. it was independent of any system parameter. Strikingly, this results originate from the momentum interval between $T^{1/3}$ and $T^{1/2}$ again.

Finally, in Sect. 5.3, the most divergent contribution in second-order perturbation theory was identified and calculated. This was done to verify, that the used RG scheme is correct in the sense that the all leading logarithmic divergencies are resummed. As a result, the second order expansion of the RG result was recovered, and no additional terms that were equally divergent appeared. However, the calculation revealed a complex interplay between the two modes, g_2 and g_3, at finite temperatures, which leads to deviations from naive scaling arguments. Although these corrections are of subleading order for the correlation length, they may play a further role for the damping of the ballistic mode.

Chapter 6

Thermodynamics

In this chapter, the thermodynamic properties of the system are discussed. Already in Chap. 3 the statement was made that the mode with the higher dynamical exponent is expected to govern the thermodynamics of the system. Since this is the Landau damped mode, the system will behave in the thermally disordered low-temperature regime according to Fermi liquid theory. Nevertheless, one can expect interesting features arising from the interplay between the two different modes. After a brief introduction of the theoretical basics of quantum phase transitions and the phase diagram the specific heat, thermal expansion and compressibility are introduced. Their behavior in the quantum critical and the low-temperature regime are obtained by naive scaling arguments. Finally, the explicit calculations for the Gaussian model are presented. While the specific heat and the thermal expansion obey the scaling results, the compressibility only agrees with the scaling result in the low temperature regime. In the quantum critical regime, the compressibility originates from the zero Matsubara mode only.

6.1   Basic Notions of Quantum Phase Transitions

Most phase transitions can be characterized by a so called *order parameter*. This order parameter vanishes in the disordered phase and becomes finite in the ordered phase. Although for some cases the order parameter is not known, every 2nd order phase transition is supposed to have one. While the expectation value of the order parameter vanishes in the disordered phase, there are of course fluctuations around zero. These fluctuations have a twofold origin; thermal fluctuations and quantum fluctuations resulting from Heisenberg uncertainty principle.

At finite temperatures thermal fluctuations are usually much stronger than quantum fluctuations in the vicinity of the transition line. Therefore finite temperature phase transitions are in general driven by thermal fluctuations and called *classical* phase transition. The situation changes if one considers systems at zero tempera-
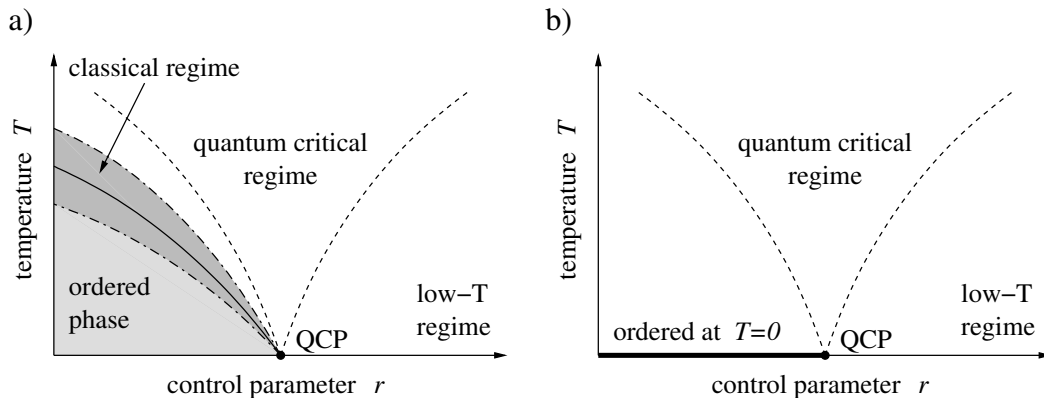
Fig. 6.1: Generic phase diagram above (a) and below (b) the lower critical dimension. The dashed lines denote the crossover from the low-temperature regime to the quantum critical regime. (Figure taken from [22].)

ture, where no thermal fluctuations are present. Depending on external parameters (e.g. pressure p or magnetic field B), there may be different ground states of the system. Tuning these parameters through a critical value yields a so called *quantum phase transition* from one into another ground state. Note that this transition is driven by quantum fluctuations only. The critical value is called the *quantum critical point*.

To make the discussion a bit more specific, one defines the distance to the critical point $r = (g - g_c)/g_c$ where g is a generic placeholder for the external parameter and g_c is the value at the quantum critical point. Close to a phase transition, fluctuations occur on all length scales and the system becomes scale invariant (or self-similar). This implies that all observables have to depend on the external parameters according to a power law. In particular, the *correlation length* ξ which is a measure for the spatial dependence of fluctuations on each other diverges according to

$$\xi \sim |r|^{-\nu} \tag{6.1}$$

The exponent ν is called the *correlation length exponent*. The decay of the critical fluctuations occur on a typical timescale τ_c which is called the *correlation time* and diverges according to

$$\tau_c \sim \xi^z \sim |r|^{-z\nu} \tag{6.2}$$

The exponent z is the dynamical exponent, as mentioned earlier in the text.

For a quantum phase transition there are two possible scenarios. Either the quantum critical point is the endpoint of a finite-temperature phase transition in a r-T-phase diagram. This possibility is depicted in Fig. 6.1 (a). For $r > 0$ the system is disordered, but tuning the parameter r through the critical value $r_c(T)$, which dependends on temperature, the system becomes ordered. At any finite temperature close to the transition line thermal fluctuations are of the order T_c and the quantum

fluctuations are of the order $\hbar\tau_c^{-1} \sim |r - r_c(T)|^{z\nu}$. Therefore in the region where $|r - r_c(T)| < T_c^{1/z\nu}$ the macroscopic critical behavior of the system is governed only by the thermal fluctuations. The shaded region around the transition line is therefore called *classical regime*.

In the other case, as depicted in Fig. 6.1 (b), there is no ordered state at any finite temperature and the quantum critical point is an isolated point of non-analytic behavior of the free energy. This is the generic case in dimensions smaller than the *lower critical dimension d_c*. The Mermin-Wagner theorem states that for $d < d_c$ thermal fluctuations prevent an ordered state at any finite temperature due to the bigger gain of entropy compared to energy. This means that the system only orders at exactly $T = 0$. At finite temperature the system is either thermally disordered or quantum disordered depending on whether the order is mainly destroyed by thermal or quantum fluctuations.

The remaining question is, for what reason other than pure academic interest these quantum phase transitions are worth to study. After all they take place at zero temperature only, which cannot be achieved in any experiment. The answer is the so called *quantum critical regime* right above the quantum critical point. The dashed crossover line is determined by the condition that the thermal fluctuations are larger than the quantum fluctuations, which means

$$k_B T > \hbar\omega_c \sim |r|^{\nu z} \qquad (6.3)$$

Here, the system is close to the critical value of the tuning parameter but thermal fluctuations drive it out of criticality. Hence, the physics are dominated by thermal excitations of the quantum critical ground state. The peculiar excitation spectrum of this state causes unusual finite-temperature behavior, like Non-Fermi liquid behavior and unconventional power laws and therefore influences observables over a wide range of the phase diagram. In particular, if a finite temperature phase transition is prevented by the Mermin-Wagner theorem, the system will nevertheless display quantum critical behavior in this region.

Having identified the different regions of the phase diagram, one notices that one can approach the quantum critical point from two different regimes. Either one chooses $|r|^{\nu z} \ll T$ which means to approach it via the quantum critical (QC) regime. One then observes the temperature scaling at the quantum critical point. Or one chooses $|r|^{\nu z} \gg T$, being in the low temperature (LT) regime, where one observes a crossover from quantum critical to classical critical behavior asymptotically close to the transition. For a more detailed discussion of quantum phase transitions see, e.g. , [23], [24] or [25].

6.2 Thermodynamic Properties

Experimentally important are the second order derivatives of the thermodynamic potential. Therefore, it is in general a good idea to calculate these thermodynamical properties and compare them to experimental data to verify a theory.

There are three possible second order derivatives of the Gibbs free energy $F(r, T)$. First of all, one can calculate the second derivative of F with respect to temperature. Since the first derivative of the free energy is the entropy, this can be rewritten as

$$c_r = -\frac{T}{V}\frac{\partial^2 F}{\partial T^2} = \frac{T}{V}\frac{\partial S}{\partial T}\bigg|_r \tag{6.4}$$

The quantity c_r is the *specific heat* at constant control parameter r, e.g. pressure.

One can also calculate the mixed derivative of F with respect to the control parameter r and temperature,

$$\alpha_r = \frac{1}{V}\frac{\partial^2 F}{\partial T \partial r} \tag{6.5}$$

For systems where the control parameter is the pressure, the derivative of F with respect to p is the volume V, and one gets

$$\alpha_p = \frac{1}{V}\frac{\partial V}{\partial T}\bigg|_p \tag{6.6}$$

which is known as the isobaric *thermal expansion* of the system.

Finally, the remaining possible second order derivative of the free energy is

$$\kappa = \frac{1}{V}\frac{\partial^2 F}{\partial r^2} \overset{r \hat{=} p}{\propto} \frac{1}{V}\frac{\partial V}{\partial p}\bigg|_T \tag{6.7}$$

where at the second equality the control parameter again was identified with pressure. This last term is known as the *compressibility* of the system.

6.3 Scaling relations

Before actually calculating the thermodynamic properties of the system, one may deduce the expected leading behavior by using the property of scale invariance at criticality. The scaling hypothesis states that at criticality there is no other characteristic length scale than the correlation length ξ. Eq. (6.1) and Eq. (6.2) then imply that upon rescaling all lengths $x \to b\,x$ the imaginary time, τ, and control parameter, r, have to be rescaled as well according to

$$\tau \to b^{-z}\tau$$
$$r \to b^{1/\nu}r \tag{6.8}$$

56

Since temperature is proportional to Matsubara frequencies it rescales like $T \to b^z T$ which implies that the free energy density scales like

$$f \to b^{z+d} f \tag{6.9}$$

Note that this is true only for models where hyperscaling applies, which is violated in theories with dangerously irrelevant variables. Such variables appear generically above the upper critical dimesion d_+ which for ϕ^4-theories is $d_+ = 4$. Right at the upper critical dimension there may appear logarithmic corrections (see also [19]).

With these scaling dimensions of the theory one can use the scale invariance of the system. After switching to a dimensionless temperature $t = T/T_0$ with a unspecified temperature scale T_0 the requirement of scale invariance implies that close to criticality the equation

$$f(r,t) = b^{-d-z} f\left(b^{1/\nu} r, b^z t\right) \tag{6.10}$$

holds. One still has the freedom of choosing the scaling parameter b.

In the quantum critical regime where $r^\nu t^{-1/z} \ll 1$ is small, one chooses $b = t^{-1/z}$ which leads to

$$
\begin{aligned}
f(r,t) = b^{-d-z} f(b^{1/\nu} r, b^z t) &= t^{1+d/z} f(t^{-1/z\nu} r, 1) \\
&= t^{d/z+1} \tilde{f}_{\mathrm{QC}}\left(t^{-1/z\nu} r\right)
\end{aligned}
\tag{6.11}
$$

Since there is no phase transition in the QC region the function $\tilde{f}_{\mathrm{QC}}$ is assumed to be analytic. Because of the smallness of $r\, t^{-1/\nu z}$, one can then expand the function as

$$\tilde{f}_{\mathrm{QC}}\left(t^{-1/z\nu} r\right) \approx \tilde{f}_{\mathrm{QC}}(0) + \tilde{f}'_{\mathrm{QC}}(0)\, t^{-1/z\nu} r \tag{6.12}$$

Note that this needs not to be true and non-analyticity is often encountered.

In the low temperature regime, where $t\, r^{-z\nu} \ll 1$ is small, one chooses $b = r^{-\nu}$. This leads to

$$
\begin{aligned}
f(r,t) = b^{-d-z} f(b^{1/\nu} r, b^z t) &= r^{\nu(z+d)} f(1, tr^{-z\nu}) \\
&= r^{\nu(z+d)} \tilde{f}_{\mathrm{LT}}\left(tr^{-z\nu}\right)
\end{aligned}
\tag{6.13}
$$

According to the third law of thermodynamics the entropy has to vanish upon approaching zero temperature. Therefore, assuming a power law behavior, one can approximate

$$\tilde{f}_{\mathrm{LT}}\left(tr^{-z\nu}\right) \approx \tilde{f}_{\mathrm{LT}}(0) + c\left(tr^{-z\nu}\right)^{y_0+1} \tag{6.14}$$

with positive constants c and y_0. These constants are in general different in the two low temperature limits, depending on whether r is smaller or larger than zero, i.e.

whether one considers the LT region on the left or the right side from the quantum critical point.

From this one can now compute the leading order dependence of the thermodynamic properties by performing the derivatives of the free energy, e.g. for the specific heat. In the quantum critical regime, one concludes that

$$c(r,t) \approx \frac{t}{T_0} \frac{\partial^2 \left(t^{d/z+1} \tilde{f}_{QC}(0) \right)}{\partial t^2} \sim \left(1 + \frac{d}{z} \right) \frac{d}{z} \, t^{d/z} \tag{6.15}$$

while in the low-temperature regime one gets

$$c(r,T) \approx \frac{t}{T_0} cr^{\nu(z+d)} \frac{\partial^2 \left(tr^{-z\nu} \right)^{y_0+1}}{\partial t^2} \sim y_0(y_0+1) \, t^{y_0} r^{\nu(d-y_0 z)} \tag{6.16}$$

The analogous calculations can be done for the thermal expansion and the compressibility which yield

$$\alpha \sim \begin{cases} \left(\frac{z+d-\frac{1}{\nu}}{z} \right) t^{(d-\frac{1}{\nu})/z} & \text{QC,} \\ \nu(d-y_0 z)(y_0+1) \, t^{y_0} r^{\nu(d-y_0 z)-1} & \text{LT,} \end{cases} \tag{6.17}$$

$$\kappa \sim \begin{cases} t^{(z+d-\frac{2}{\nu})/z} & \text{QC,} \\ (z+d-\frac{1}{\nu})(z+d) \, r^{\nu(d+z)-2} & \text{LT.} \end{cases}$$

For further introductions on scaling theory see also [25], [26].

6.4 Thermodynamic Calculations

In this section the behavior of the system is calculated and the results are compared to the ones obtained from the scaling ansatz. First of all, one calculates the *free energy density*, f, which is defined as

$$f = \frac{F}{V} = -\frac{1}{V} T \ln \mathcal{Z} \tag{6.18}$$

In a perturbative approach, one expands the partition function $\mathcal{Z}$ in the following way

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-S_0 - S_{\text{int}}} = \int \mathcal{D}\phi \, e^{-S_0} \sum_{n=0}^{\infty} \frac{1}{n!} (-S_{\text{int}})^n$$

$$= \mathcal{Z}_0 + \mathcal{Z}_0 \sum_{n=1}^{\infty} \frac{1}{n!} \langle (-S_{\text{int}})^n \rangle_0 \tag{6.19}$$

$$\mathcal{Z}_0 = \int \mathcal{D}\phi \, e^{-S_0}$$

Here, S_0 is the quadratic part of the action, and the average in the second line is taken with respect to this part only. Substituting this result into Eq. (6.18) yields

$$f = -\frac{T}{V} \ln \mathcal{Z}_0 - \frac{T}{V} \ln \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \langle (-S_{\text{int}})^n \rangle_0 \right) \tag{6.20}$$

which to first order in the coupling constant u_0 takes the form

$$
\begin{aligned}
f &\approx \frac{T}{V} \ln \mathcal{Z}_0 + \frac{T}{V} \langle S_{\text{int}} \rangle_0 + \mathcal{O}(u^2) \\
&= \frac{T}{V} \ln \left\{ \left[\operatorname{Tr} g^{-1} \right]^{1/2} \right\} + \frac{u_0}{4!} \left(\frac{T}{V} \right)^3 \sum_{k,q,p} \left\langle n_k^{\text{i}} n_q^{\text{i}} n_p^{\text{j}} n_{-k-q-p}^{\text{j}} \right\rangle_0
\end{aligned}
\tag{6.21}
$$

Here, the summation over the indices $\{i, j\}$ is implicit. On a Gaussian level, one considers the first term only which can be rewritten as

$$
\begin{aligned}
f_0 &= \frac{T}{V} \ln \left\{ \left[\operatorname{Tr} g^{-1} \right]^{1/2} \right\} = \frac{1}{2} \frac{T}{V} \sum_q (\ln g_2^{-1}(q) + \ln g_3^{-1}(q)) \\
&= \frac{1}{2} \int \frac{\mathrm{d}^d q}{(2\pi)^2} T \sum_{\omega_n} \left[\ln \left(r + q^2 + \frac{\omega_n^2}{q^2} \right) + \ln \left(r + q^2 + \gamma_0 \frac{|\omega_n|}{q} \right) \right]
\end{aligned}
\tag{6.22}
$$

Converting the Matsubara frequency summation into a contour integration, as it was done in the preceding chapter, and after some algebra one gets

$$
f_0 = K_2 \int_0^\Lambda \mathrm{d}q\, q \left[T \ln \left(\sinh \frac{\omega_q}{2T} \right) + \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \coth \frac{\omega}{2T} \arctan \frac{\gamma_0 \omega q}{\omega_q^2} \right]
\tag{6.23}
$$

where $\omega_q = q(r + q^2)^{1/2}$ as before. The first term originates from the g_2 mode, while the second term, containing the frequency integration, comes from the g_3 mode.

The first-order correction can actually be rewritten as a derivative of the Gaussian free energy. It is

$$
\begin{aligned}
f_1 &= \frac{2u_0}{4!} \left(\frac{T}{V} \right)^2 \sum_{k,q} (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{mj} + \delta_{im} \delta_{lj})\, g^{ij}(k) g^{lm}(q) \\
&= \frac{2u_0}{4!} \left[\frac{T}{V} \sum_q g_2(q) + g_3(q) \right]^2 = \frac{2u_0}{4!} \left(\frac{\partial f_0}{\partial r} \right)^2
\end{aligned}
\tag{6.24}
$$

Due to this relation the first order corrections can be calculated derived from the results of the Gaussian calculation and one can compute all three thermodynamic quantities introduced above.

However, before this is done, one has to recall the phase diagram of this special problem. Since the two modes, g_2 and g_3 have different dynamical exponents, the phase diagram consist not only of two but of three different regions (see Fig. 6.2). For temperatures $T > |r|$ the system is in the quantum critical regime of both modes, and for temperatures $T < |r|^{3/2}$ the system is in the low temperature regime of both modes. In the region $|r| > T > |r|^{3/2}$ the low temperature regime of the g_2 mode overlaps with the quantum critical regime of the g_3 mode. In this overlap (OL) regime, one has to compare the low temperature result of the g_2 mode with the quantum critical result of the g_3 mode.
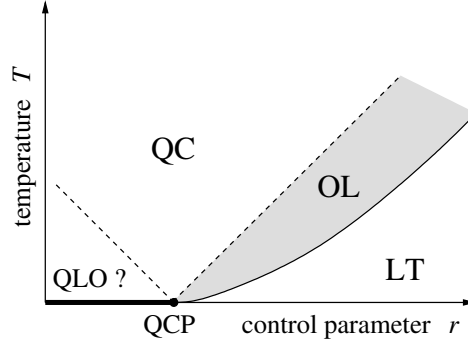
Fig. 6.2: Different regions in the phase diagram of the system. There is the region above the quantum critical point where both modes are in their quantum critical regime (QC) and is the low temperature regime of both modes (LT). Additionally, there is a overlap regime (OL), where the g_2 mode is already in its LT regime while the g_3 mode is still in its QC regime. The thermally disordered region (QLO) above the ordered ground state is not exactly known. However, one can expect a state of quasi-long ranged order.

6.4.1 Specific Heat

To obtain the specific heat, one differentiates Eq. (6.23) twice with respect to temperature and multiplies by T. Split into the contributions coming from the two modes respectively one obtains

$$c_2^0 = \frac{K_2}{4T^2} \int_0^\Lambda \mathrm{d}q\, q\, \omega_q^2 \sinh^{-2}\left(\frac{\omega_q}{2T}\right) \tag{6.25}$$

$$c_3^0 = \frac{K_2}{T} \int_0^\Lambda \mathrm{d}q\, q \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{\omega}{2T} \sinh^{-2}\left(\frac{\omega}{2T}\right)\left(\frac{\omega}{2T}\coth\frac{\omega}{2T} - 1\right)\arctan\frac{\gamma_0\omega q}{\omega_q^2} \tag{6.26}$$

In order to analyze the two contributions, one may rescale momentum and frequency in both expressions. For the g_2 mode one rescales $q^2 \to 2Tq^2$ and introduces $\bar\omega_q = q(\frac{r}{2T} + q^2)^{1/2}$ which leads to

$$c_2^0 = 2TK_2 \int_0^{\Lambda/\sqrt{2T}} \mathrm{d}q\, q\, \bar\omega_q^2 \sinh^{-2}\bar\omega_q \tag{6.27}$$

For the g_3 mode one has to rescale frequencies according to $\omega \to 2T\omega$ and momenta $q^3 \to 2Tq^3$. Introducing further $\tilde\omega_q = q\left(\frac{r}{(2T)^{2/3}} + q^2\right)^{1/2}$ one obtains

$$c_3^0 = 2K_2(2T)^{2/3} \int_0^{\Lambda/(2T)^{1/3}} \mathrm{d}q\, q \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega \sinh^{-2}\omega\,(\omega\coth\omega - 1)\arctan\frac{\gamma_0\omega}{\tilde\omega_q^2} \tag{6.28}$$

In the quantum critical regime, a careful analysis shows that both contributions are independent of the momentum cutoff. Moreover, since the region in momentum space between zero and $\frac{r}{2T}$ or $\frac{r}{(2T)^{2/3}}$ is negligible small, one can approximate $\omega_q \approx q^2$ over the whole range of integration. One therefore concludes that c_2^0 grows linearly

with temperature while c_3^0 grows according to $T^{2/3}$.

$$c_{QC}^0 \sim \begin{cases} T & \text{for } g_2 \\ T^{2/3} & \text{for } g_3 \end{cases} \qquad (6.29)$$

This result is in perfect agreement with the one obtained by the scaling approach in two dimensions.

In the low-temperature regime, the value of $\frac{r}{2T}$ and $\frac{r}{(2T)^{2/3}}$ is relatively large. For momenta smaller than these ratios, one can approximate $\omega_q \approx q\sqrt{r}$. This implies that in the low-momentum regime on has to rescale momenta differently. For the ballistic mode with $z = 2$ one chooses $q^2 \to \frac{4T^2}{r}q^2$, while for the damped mode one rescales according to $q \to \frac{2T}{r}q$. A careful analysis of the two integrations leads to the result that for both modes this lower momentum regime gives the leading order behavior. One obtains the specific heat in the low temperature regime to be

$$c_{LT}^0 \sim \begin{cases} T^2 r^{-1} & \text{for } g_2 \\ T r^{-1/2} & \text{for } g_3 \end{cases} \qquad (6.30)$$

Again the results are in agreement with the one obtained by the scaling approach in two dimensions. However, while the damped mode has an exponent $y_0 = 1$ which is characteristic for a Fermi liquid, one has to set $y_0 = 2$ for the ballistic mode. The Non-Fermi liquid behavior of the g_2 mode is not such a surprise since the important particle-hole excitations are excluded and Landau damping is not active. Finally, the specific heat in the overlap regime is obtained to be

$$c_{OL}^0 \sim \begin{cases} T^2 r^{-1} & \text{for } g_2 \\ T^{2/3} & \text{for } g_3 \end{cases} \qquad (6.31)$$

To find the leading order behavior of the specific heat of the whole system, one has to compare the two contributions in each regime respectively. One finds that the g_3 mode governs the specific heat in the quantum critical regime, as well as in the low temperature regime. For the overlap regime of the phase diagram holds $r^{3/2} < T < r < r^{4/3}$, and therefore, the specific heat is governed by the g_3 mode also in this regime. One concludes that the whole system shows ($z = 3$) behavior over the whole phase diagram. There is one crossover line from the overlap regime to the low temperature regime

$$c^0 \sim \begin{cases} T^{2/3} & \text{in QC regime} \\ T^{2/3} & \text{in OL regime} \\ \frac{T}{\sqrt{r}} & \text{in LT regime} \end{cases} \qquad (6.32)$$

The inclusion of finite interactions yields logarithmic correction does. This is, because one essentially has to replace r by the correlation length $\xi^{-2} \sim r/(\ln(1/r))^{4/9}$ obtained in Sect. 4.2.

6.4.2 Thermal Expansion

Differentiating Eq. (6.23) with respect to the control parameter r leads to the very same expression like the Hartree diagram considered in Chap 5. This has to be differentiated with respect to temperature and yields for the both modes

$$\alpha_2^0 = \frac{K_2}{4T^2} \int_0^\Lambda dq \, \frac{q^3}{2} \sinh^{-2}\left(\frac{\omega_q}{2T}\right) \tag{6.33}$$

$$\alpha_3^0 = \frac{K_2}{4T^2} \int_0^\Lambda dq \, q \int_0^\infty \frac{d\omega}{\pi} \sinh^{-2}\left(\frac{\omega}{2T}\right) \frac{q\gamma_0\omega^2}{q^{-2}\omega_q^4 + \gamma_0^2\omega^2} \tag{6.34}$$

In the quantum critical region, one performs the same rescaling procedure as done for the specific heat, approximating again $\omega_q \approx q^2$. However, one has to cut off the momentum integration at a scale $\sqrt{r/(2T)}$ for the ballistic mode and at a scale $r^{1/2}/(2T)^{2/3}$ for the damped mode. These integrals are logarithmically divergent in temperature for both modes

$$\alpha_{QC}^0 \sim \begin{cases} \ln(cr/T) & \text{for } g_2 \\ \ln(c'T/r^{3/2}) & \text{for } g_3 \end{cases} \tag{6.35}$$

The constants c and c' are left unspecified since they are of no concern. This result is in agreement with the scaling approach prediction, since it states a temperature exponent of zero for both modes in the quantum critical region.

In the low-temperature regime one may again split the momentum integration in the two regimes where one can approximate ω_q either by $q\sqrt{r}$ or by q^2. The rescaling is done exactly like for the specific heat, and the resulting integrals are obtained to scale like

$$\alpha_{LT}^0 \sim \begin{cases} T^2 \, r^{-2} & \text{for } g_2 \\ T \, r^{-3/2} & \text{for } g_3 \end{cases} \tag{6.36}$$

The g_3 mode agrees with the scaling result for an exponent $y_0 = 1$, i.e. showing Fermi liquid behavior. The g_2 mode also gives the expected result for $y_0 = 2$. This result is also in agreement with the calculation of the specific heat.

In the overlap regime, the contributions to the thermal expansion of both modes are, respectively

$$\alpha_{OL}^0 \sim \begin{cases} T^2 \, r^{-2} & \text{for } g_2 \\ \ln(c'T/r^{3/2}) & \text{for } g_3 \end{cases} \tag{6.37}$$

Comparing both contributions in the two coinciding regimes, one obtains that the thermal expansion of the whole system depends on temperature and control parameter according to the g_3 mode. In the overlap regime, the contribution of the g_2 mode decrease from a constant of order one to r when lowering the temperature from $T = r$ to $T = r^{3/2}$. At the same time the g_3 mode decrease from $\ln r$ to a constant.

Thus, the g_3 mode is always dominant in the overlap regime and therefore dominates the whole phase diagram in terms of the thermal expansion. To summarize, the behavior of the thermal expansion is

$$\alpha^0 \sim \begin{cases} \ln(c'T/r^{3/2}) & \text{in QC regime} \\ \ln(c'T/r^{3/2}) & \text{in OL regime} \\ T\, r^{-3/2} & \text{in LT regime} \end{cases} \tag{6.38}$$

As for the specific heat, one crossover line from the overlap regime to the low temperature regime can be seen.

The calculation of the corrections is not so easy as for the specific heat. This is, because for the thermal expansion one has to differentiate with respect to r. It is not obvious whether a replacement of r by $\xi(r)^{-2}$ is sufficient to obtain the correct result. However, logarithmic corrections are expected, but the calculations are, unfortunately, not done, yet. At least, one can see that in the quantum critical regime, where it is $r \sim T$, the thermal expansion shows a logarithmic temperature dependence as it was expected.

6.4.3 Compressibility

Finally, one may calculate the compressibility by differentiating Eq. (6.23) twice with respect to r. The resulting integral expressions of the contribution of the modes reads

$$\kappa_2^0 = -K_d \int_0^\Lambda \mathrm{d}q\, \frac{q^5}{8\omega_q^3} \left[\frac{\omega_q}{2T} \sinh^{-2}\left(\frac{\omega_q}{2T} \right) + \coth \frac{\omega_q}{2T} \right] \tag{6.39}$$

$$\kappa_3^0 = -K_d \int_0^\Lambda \mathrm{d}q \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \coth \frac{\omega}{2T} \frac{2q^2\gamma_0\omega\,\omega_q^2}{\left(q^{-2}\omega_q^4 + \gamma_0^2\omega^2\right)^2} \tag{6.40}$$

A careful analysis (done by the same means as used for specific heat and thermal expansion) leads to the conclusion that the g_2 mode is logarithmic UV-divergent while the g_3 mode depends linear on the cutoff. These UV divergencies can be interpreted as a non-universal background which has to be subtracted. Doing so, one obtains the compressibility in the quantum critical regime as

$$\kappa_{QC}^0 \sim \begin{cases} T/r & \text{for } g_2 \\ T/r & \text{for } g_3 \end{cases} \tag{6.41}$$

These contributions originate from the zero Matsubara modes only and obviously diverge for $r \to 0$. The Gaussian level, therefore, is not appropriate to obtain the compressibility in the quantum critical regime, but one has to take finite interactions into account.

In the low temperature regime, applying the same methods as before, one obtains

$$\kappa_{LT}^0 \sim \begin{cases} \ln(c\, r/\Lambda^2) & \text{for } g_2 \\ \sqrt{r} & \text{for } g_3 \end{cases} \tag{6.42}$$

The logarithmic cutoff dependence of the g_2 mode signals an IR divergence. This implies that the Gaussian level is not appropriate in the low temperature regime either. One rather has to sum up these IR divergences in any order of perturbation theory. This can be done by deriving a RG equation for the compressibility. In particular, a plain substitiution of r by ξ^{-2}, which may work for the g_3 mode, is not sufficient.

In the overlap regime, where g_2 is in its low temperature phase and g_3 is in its quantum critical phase, the compressibility is

$$\kappa^0_{OL} \sim \begin{cases} \ln(c\,r/\Lambda^2) & \text{for } g_2 \\ T/r & \text{for } g_3 \end{cases} \tag{6.43}$$

Comparison of both contributions yields a dominating behavior of the g_2 mode in the over the whole phase diagram.

$$\kappa_0 \sim \begin{cases} T/r & \text{in QC regime} \\ \ln(c\,r/\Lambda^2) & \text{in OL regime} \\ \ln(c\,r/\Lambda^2) & \text{in LT regime} \end{cases} \tag{6.44}$$

Here, the crossover takes place from the quantum critical to the overlap regime.

The results of this calculation indicate that one has to take finite interactions seriously into account. As for the thermal expansion, a pure replacement of r by $\xi(r)^{-2}$ in Eq. (6.23) may be not sufficient because of the twofold derivative with respect to r. Rather one has to do an infinite resummation by means of RG methods.

A comparison with the result obtained by pure scaling arguments shows an agreement in the low temperature regime. In the quantum critical regime the assumption of an analytical scaling function, f_{QC}, leads to $\kappa \sim T^0$ for the g_2 mode and $\kappa \sim T^{1/3}$ for the g_3 mode. However, if one uses that $r \sim T$, one recovers the scaling result at least for the g_2 mode. This illustrates the necessity of taking interactions into account.

6.5 Summary

The calculations, done in this chapter, show that the mode with the larger dynamical exponent, g_3, indeed governs up to logarithmic corrections the specific heat and the thermal expansion of the system. The results are in agreement with the behavior obtained from pure scaling arguments. Furthermore, the leading g_3 mode shows in the low temperature regime away from the quantum critical point Fermi liquid behavior, having an exponent $y_0 = 1$. In contrast, the subleading g_2 mode has an exponent $y_0 = 2$ implying Non-Fermi liquid behavior. Such an exponent also appears, e.g. , for d-wave superconductors and may originate from the quadrupolar interaction of the electrons. In summary, the whole system shows, in the isotropic state, Fermi liquid behavior.

The compressibility is in the low temperature regime also in agreement with the prediction of the scaling ansatz. Strikingly, the compressibility is governed by the mode with the lower dynamical exponent, i.e. the g_2 mode. In particular, this result contradicts with the statement of Oganesyan et al. [1], which claimed the $z = 2$ mode to be irrelevant.

The result, obtained on a strictly Gaussian level, are not satisfying since interactions were neglected. Especially the diverging compressibility is an artifact of this approximation. As soon as one takes finite interactions into account, the divergence is prevented. The thermal expansion gets finite interaction corrections for the same reason.

Chapter 7

Summary and Outlook

In this thesis, the transition from a isotropic to a nematic fermionic system in spatial dimensions $d = 2$ was studied. Starting from free fermions subject to a quadrupolar interaction, the effective Ginzburg-Landau-Wilson action for the order parameter was derived in Chap. 3. A peculiarity of this action is the existence of two modes which become simultaneously critical but have different dynamics. While the g_3 mode is subject to Landau damping and has a dynamical critical exponent of $z = 3$, the g_2 mode is ballistic and has a dynamical exponent $z = 2$.

An investigation at zero temperature by means of renormalization group methods required an adjusting of the usual scheme due to the two different dynamical exponents. Within the scheme delivered in Chap. 4, a RG analysis yields that at large scales the correlation length is determined by the g_2 mode only. This is in contrast to the scaling arguments given by Oganesyan et. al [1], which state that only the g_3 mode is important for the critical behavior. Moreover, because of the interactions between both modes, the system was obtained being neither of the Ising universality class nor of the universality class of the XY model.

At finite temperatures the intricate interplay of the two modes leads to interesting consequences. As shown in Chap. 5, the model is governed by the momentum interval $k \in [T^{1/3}, T^{1/3}]$. Here, the g_3 mode behaves as in a classical d-dimensional field theory while the g_2 mode is still controlled by its quantum character and has an effective dimension $d + z = 4$. The former is, therefore, at the lower critical dimension of the theory while the latter is at its upper critical dimension.

The existence of this crossover regime is the reason that the theory can not be reduced to a d-dimensional effective theory for the zero Matsubara modes, implying a breakdown of dimensional reduction. It also leads to a correlation length which is, to leading order, linear in temperature. Moreover, the prefactor is obtained to be independent of the interaction strength. In Sect. 5.3 it was furthermore shown that the interplay of the two modes, g_2 and g_3, can lead to completely unexpected corrections, although for the concrete example the obtained correction was only

subleading.

Finally, in Chap. 6 the thermodynamic quantities specific heat, thermal expansion and compressibility were calculated on a Gaussian level. As expected, thermodynamics are governed by the g_3 mode, due to the larger available phase space. The expressions for the specific heat and the thermal expansion agree with the naive scaling results. However, the compressibility turns out to have a finite cutoff dependend background contribution and is, to first order, determined by the zero Matsubara mode only. The arising divergencies above the quantum critical point as well as the logarithmic cutoff dependence of the g_2 mode indicate the incompleteness of the Gaussian approximation and the necessity of interactions.

As a next step, it is necessary to calculate the interaction corrections to the compressibility and the thermal expansion. To resum the arising divergencies one has to use the developed RG-sheme again. It would also be interesting to calculate the lifetime of the ballistic mode, g_2. A further expansion of the dynamical part of g_2 in ω/q gives rise to the damping term $|\omega|^3/q^3$. In addition, the interaction with the g_2 mode as well as with the g_3 mode leads to another damping contribution. Therefore the interesting interplay between both modes discovered in Sect. 5.3 may become very important.

Appendix A

Derivation of the Effective Action

Starting from the action

$$S[\bar{\psi}, \psi, n] = -\frac{1}{2f_2} \int dx\, d\tau\, n^2 - \mathrm{Tr}\left\{ \ln\left(\mathcal{G}_0^{-1} + n \cdot \Delta \right) \right\}$$

$$= -\frac{1}{2f_2} \int dx\, d\tau\, n^2 + S_0 - \mathrm{Tr}\left\{ \mathcal{G}_0 n \cdot \Delta \right\} + \frac{1}{2} \mathrm{Tr}\left\{ (\mathcal{G}_0 n \cdot \Delta)(\mathcal{G}_0 n \cdot \Delta) \right\} + \ldots$$

established in Chap. 3, one first of all drops the constant fermionic part of the action, S_0. The calculation of the two terms $\mathrm{Tr}\left\{ \mathcal{G}_0 n \cdot \Delta \right\}$ and $\mathrm{Tr}\{(\mathcal{G}_0 n \cdot \Delta)^2\}$ is most conveniently done in momentum representation.

The first trace has to vanish due to the fact that the original order parameter is a traceless matrix. In fact, since Δ and $\mathcal{G}$ are both diagonal in momentum space, we obtain

$$\mathrm{Tr}\left\{ \mathcal{G}_0 n \cdot \Delta \right\} = \frac{T}{L^d} \sum_k \mathcal{G}_{0,k} n_0^i \Delta_k^i = n_0^i \frac{1}{L^d} \sum_{\mathbf{k}} n_F(\xi_{\mathbf{k}}) \Delta_{\mathbf{k}}^i$$

$$= n_0^i \frac{1}{k_F^2} \int \frac{d^2k}{(2\pi)^2} n_F(\xi_k) \begin{pmatrix} \cos 2\phi \\ \sin 2\phi \end{pmatrix} = 0$$

In contrast, the second term will not vanish but yield additional terms quadratic in the field n.

$$\mathrm{Tr}\{(\mathcal{G}_0 n \cdot \Delta)^2\} = \left(\frac{T}{L^d} \right)^2 \sum_{k,k'} \mathcal{G}_{0,k} n_{k-k'}^i \Delta_{k'}^i \mathcal{G}_{0,k'} n_{k'-k}^j \Delta_k^j$$

$$= \left(\frac{T}{L^d} \right)^2 \sum_q n_q^i n_{-q}^j \sum_k \mathcal{G}_{0,k} \Delta_{k-q}^i \mathcal{G}_{0,k-q} \Delta_k^j = \frac{T}{L^d} \sum_q n_q^i \Pi_q^{ij} n_{-q}^j$$

$$(\text{A.1})$$

The dynamical and the static part of the tensor Π^{ij} have to be calculated separately. Starting with the dynamical part, one can neglect the q-dependence of Δ for small

momenta

$$\Pi^{ij}(\mathbf{q}, \Omega) = \frac{T}{L^d} \sum_k \mathcal{G}_{0,k+q} \Delta_k^i \mathcal{G}_{0,k} \Delta_{k+q}^j \tag{A.2}$$

$$\approx \frac{T}{L^d} \sum_{\mathbf{k},\omega} \frac{1}{i\omega - \xi_{\mathbf{k}}} \frac{1}{i\omega + i\Omega - \xi_{\mathbf{k+q}}} \frac{k^4}{k_F^4} \begin{pmatrix} \cos^2(2\phi) & \cos(2\phi)\sin(2\phi) \\ \cos(2\phi)\sin(2\phi) & \sin^2(2\phi) \end{pmatrix}^{i.j}$$

Here, ϕ is the angle between $\mathbf{k}$ and the x axis.

Choosing a frame of reference in which $\mathbf{q}$ is parallel to the x-axis one can expand $\xi_{\mathbf{k+q}}$ to first order in $\mathbf{q}$ yielding $\xi_{\mathbf{k+q}} = \xi_{\mathbf{k}} + v_F q \cos\phi$, where v_F is the Fermi velocity. Note that this implies that in Eq. (A.1) the frame of reference for each summation term is a different one, and n_q is rotated.

Changing from summation to integration and approximating $k^5/k_F^4 \approx k$ one arrives at the integral

$$\Pi(\mathbf{q}, \Omega) = \int \frac{d\omega}{2\pi} \int dk \int_0^\pi \frac{d\phi}{(2\pi)^2} \frac{k}{i\omega - \xi_{\mathbf{k}}} \left[\frac{1}{i\omega + i\Omega - \xi_{\mathbf{k}} - v_F q \cos\phi} + \frac{1}{i\omega + i\Omega - \xi_{\mathbf{k}} + v_F q \cos\phi} \right]$$

$$\times \begin{pmatrix} (\cos^2\phi - \sin^2\phi)^2 & 2\cos\phi\sin\phi(\cos^2\phi - \sin^2\phi) \\ 2\cos\phi\sin\phi(\cos^2\phi - \sin^2\phi) & 4\cos^2\phi\sin^2\phi \end{pmatrix}$$

Upon substituting $d\xi_{\mathbf{k}} = 2\pi k/\nu$ with the density of states $\nu = m^\star/(2\pi)$ according to Eq. (2.8), one can simplify this equation to

$$\Pi(\mathbf{q}, \Omega) = \int \frac{d\omega}{(2\pi)^2} \int d\xi \frac{\nu}{i\omega - \xi} \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} \left[\frac{1}{i\omega + i\Omega - \xi - v_F q x} + \frac{1}{i\omega + i\Omega - \xi + v_F q x} \right]$$

$$\times \begin{pmatrix} 4x^4 - 4x^2 + 1 & 2x\sqrt{1 - x^2}(2x^2 - 1) \\ 2x\sqrt{1 - x^2}(2x^2 - 1) & 4x^2 - 4x^4 \end{pmatrix}$$

where it was substituted further $x = \cos\phi$. Since the term in brackets is symmetric with respect to $x \to -x$, the antisymmetric off-diagonal terms of Π will vanish upon integration over x, so one can drop them from the beginning.

To perform the integration over ξ one applies the residue theorem, by extending the integration from $[-\infty, \infty]$ to a semi-circle in the upper complex plane (see Fig. A.1). The function has, depending on Ω, up to two simple poles in the upper complex plane, which lead to the following result

$$\Pi = i\nu \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} \left[\frac{1}{i\Omega - v_F q x} + \frac{1}{i\Omega + v_F q x} \right] \begin{pmatrix} 4x^4 - 4x^2 + 1 & \\ & 4x^2 - 4x^4 \end{pmatrix}$$

$$\times \int \frac{d\omega}{2\pi} (\Theta(\omega) - \Theta(\omega + \Omega))$$

$$= i\nu \frac{\Omega}{2\pi} \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} \left[\frac{1}{i\Omega - v_F q x} + \frac{1}{i\Omega + v_F q x} \right] \begin{pmatrix} 4x^4 - 4x^2 + 1 & \\ & 4x^2 - 4x^4 \end{pmatrix}$$
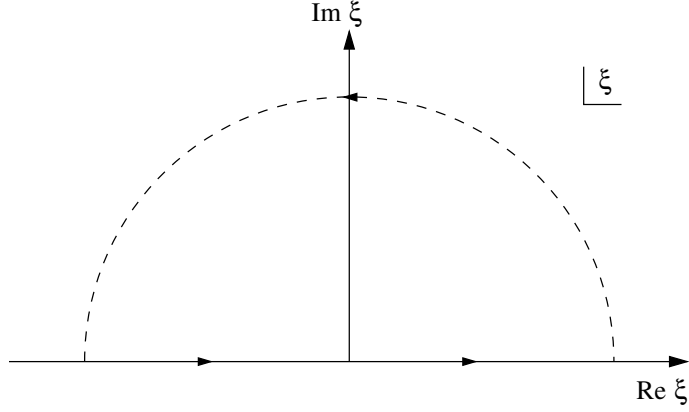
Fig. A.1: Picture of the integration contour for the ξ integration. The integration over the dashed semi-circle gives no contribution since the integration kernel vanishes faster than ξ^{-1} for $\xi \to \infty$. On the other hand, the closed contour permits the use of the residue theorem to perform the integral.

With $s = i\Omega/(v_F q)$ this relation takes the form

$$\Pi = \frac{\nu}{2\pi} s \begin{pmatrix} 4I_4 - 4I_2 + I_0 & \\ & 4I_2 - 4I_4 \end{pmatrix} \tag{A.3}$$

and one has to deal with integrals of the form

$$I_n = \int_{-1}^{1} \mathrm{d}x \frac{x^n}{\sqrt{1-x^2}} \left(\frac{1}{s+x} + \frac{1}{s-x} \right) = \int_{-1}^{1} \mathrm{d}x \frac{x^n}{\sqrt{1-x^2}} \frac{2s}{s^2 - x^2}$$

These integrals obey the following relations

$$I_4 = s^2 I_2 - \pi s$$
$$I_2 = s^2 I_0 - 2\pi s$$

and one only has to calculate I_0. To carry out the integral I_0, one first shifts $x \to x + i\epsilon$ by an infinitesimal positive ϵ. After this shift, one can continue the integration to $\pm\infty$, since the additional contributions cancel each other in the limit of $\epsilon \to 0^+$. For $\sigma = \pm 1$ denoting the positive and negative real axis, these contributions are given by

$$\sigma \int_{\sigma 1}^{\sigma\infty} \frac{\mathrm{d}x}{\sqrt{1-(x+i\epsilon)^2}} \frac{2s}{s^2 - (x+i\epsilon)^2} = \int_{\infty}^{1} \frac{\mathrm{d}x}{\sqrt{1-(x+i\sigma\epsilon)^2}} \frac{2s}{s^2 - (x+i\sigma\epsilon)^2}$$

where the substitution $x \to \sigma x$ was made. Since s is purely imaginary, ϵ may be safely sent to zero in the second term while the squareroot is expanded to

$$\sqrt{1-(x+i\sigma\epsilon)^2} = \exp\left[\frac{1}{2} \ln\left(1-(x+i\sigma\epsilon)^2\right) \right]$$
$$= \exp\left[\frac{1}{2} \left\{ \ln(x^2-1) + \ln\left(-1 + \frac{2i\sigma x\epsilon}{x^2-1} \right) \right\} \right]$$
$$= \sqrt{x^2-1} \, e^{i \, \mathrm{sgn}(\sigma x)\pi/2}$$

where in the last line the branch cut of the second logarithm at the negative x-axis yields π times the sign of the imaginary part. The remainig exponential gives $-i\sigma$, and the integrals may be written as

$$i\sigma \int_{\infty}^{1} \frac{\mathrm{d}x}{\sqrt{x^2-1}} \frac{2s}{s^2-x^2}$$

These cancel indeed each other. The integral over the entire real x-axis may now be extended by a semi-circle in the upper complex plain which gives no contribution, due to the integrand vanishing faster than $1/x$. Hence, one can use the residue theorem and sum up the pole $|s|$ in the upper complex plain which finally gives

$$I_0 = -\frac{2\pi i}{\sqrt{1-s^2}} \, \mathrm{sgn}(s)$$

Substituting this result into Eq. (A.3) yields to leading order in $|s|$

$$\Pi(\mathbf{q}, \Omega) = \nu \begin{pmatrix} |s| - 2s^2 \\ & 2s^2 - 4|s|^3 \end{pmatrix} \approx \nu \begin{pmatrix} \frac{|\Omega|}{v_F q} \\ & 2\left(\frac{\Omega}{v_F q}\right)^2 \end{pmatrix}$$

For the static contribution, $\Pi(\mathbf{q}, 0)$ one has to treat the momentum dependence more carfully. In particular, one must not drop the q-dependence of Δ_{k+q} in Eq. A.2. Choosing again the frame of reference such that $\mathbf{q} = q\,\mathbf{e}_x$ one gets

$$\Pi^{ij}(\mathbf{q}, \Omega) = \frac{T}{L^d} \sum_k \mathcal{G}_{0,k+q}\Delta_k^i \mathcal{G}_{0,k}\Delta_{k+q}^j$$

$$= \frac{T}{k_F^4 L^d} \sum_{\mathbf{k},\omega} \frac{1}{i\omega - \xi_{\mathbf{k}}} \frac{1}{i\omega + i\Omega - \xi_{\mathbf{k+q}}} \begin{pmatrix} (k_x^2 - k_y^2)((k_x+q)^2 - k_y^2) \\ & 4k_x k_y^2(k_x+q) \end{pmatrix}_{ij}$$

$$= \frac{1}{k_F^4 L^d} \sum_{\mathbf{k}} \frac{n_f(\epsilon_k) - n_f(\epsilon_{k+q})}{i\Omega - \xi_{\mathbf{k+q}} + \xi_{\mathbf{k}}} \begin{pmatrix} (k_x^2 - k_y^2)((k_x+q)^2 - k_y^2) \\ & 4k_x k_y^2(k_x+q) \end{pmatrix}_{ij}$$

Shifting the momentum integration such that the Fermi functions $n_F(\epsilon)$ have the same argument, and sending $\Omega \to 0$ to obtain the static contribution one gets

$$\Pi(\mathbf{q}, 0) = \frac{1}{k_F^4} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \left[\frac{n_f(\epsilon_k)}{\xi_{\mathbf{k}} - \xi_{\mathbf{k+q}}} \begin{pmatrix} (k_x^2 - k_y^2)((k_x+q)^2 - k_y^2) \\ & 4k_x k_y^2(k_x+q) \end{pmatrix} \right.$$

$$\left. - \frac{n_f(\epsilon_k)}{\xi_{\mathbf{k-q}} - \xi_{\mathbf{k}}} \begin{pmatrix} (k_x^2 - k_y^2)((k_x-q)^2 - k_y^2) \\ & 4k_x k_y^2(k_x-q) \end{pmatrix} \right]$$

In the zero temperature limit, the Fermi functions become Heaviside step functions, which restrict the momentum integration to the Fermi sphere. Together with the dispersion relation $\xi_{\mathbf{k+q}} = \frac{1}{2m_\star}\left[(k_x+q)^2 + k_y^2\right]$ one then obtains

$$\Pi(\mathbf{q}, 0) = \frac{2m_\star}{k_F^4} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \Theta(-\xi_{\mathbf{k}}) \left\{ \left[\frac{1}{2qk_x - q^2} - \frac{1}{2qk_x + q^2} \right] \begin{pmatrix} (k_x^2 - k_y^2)(k_x^2 - k_y^2 + q^2) \\ & 4k_x^2 k_y^2 \end{pmatrix} \right.$$

$$\left. + \left[\frac{1}{2qk_x - q^2} + \frac{1}{2qk_x + q^2} \right] \begin{pmatrix} 2k_x q(k_x^2 - k_y^2) \\ & 4k_x k_y^2 q \end{pmatrix} \right\}$$

Switching to polar coordinates and performing the ϕ-integral yields

$$\Pi(\mathbf{q},0) = \frac{\nu}{\pi k_F^4 q} \int_0^{k_F} \mathrm{d}k \; k^3 \begin{pmatrix} \pi(q^2/k^2 - 2) & \\ & \pi(q^2/k^2 - 2) \end{pmatrix}$$

The remaining momentum integration is easily carried out

$$\Pi^{1,1}(\mathbf{q},0) = \Pi^{2,2}(\mathbf{q},0) = -\frac{\nu}{2}\left(1 - \frac{q^2}{k_F^2}\right) + \mathcal{O}\left(\frac{q^4}{k_F^4}\right)$$

Adding static and dynamical contributions as well as the original quadratic term, the inverse propagator for the field n reads

$$
\frac{1}{2}g_0^{-1}(\mathbf{q},\Omega) = -\frac{1}{2f_2} + \frac{1}{2}\Pi(\mathbf{q},\Omega)
$$
$$
= \frac{\nu}{2}\begin{pmatrix} \left(-\frac{1}{\nu f_2} - \frac{1}{2}\right) + \frac{q^2}{2k_F^2} + \frac{|\Omega|}{v_F q} & \\ & \left(-\frac{1}{\nu f_2} - \frac{1}{2}\right) + \frac{q^2}{2k_F^2} + 2\left(\frac{\Omega}{v_F q}\right)^2 \end{pmatrix}
$$

Setting $\nu = 1$ and rescaling momenta and frequencies according to $q^2/(2k_F^2) \to q^2$ and $\Omega^2/(2v_F^2 k_F^2) \to \Omega^2$ we finally get

$$
g_0^{-1}(\mathbf{q},\Omega) = \begin{pmatrix} r_0 + \frac{q^2}{2k_F^2} + \gamma_0\frac{|\Omega|}{q} & \\ & r_0 + q^2 + \left(\frac{\Omega}{q}\right)^2 \end{pmatrix}
$$

where $r_0 = \left(-\frac{1}{\nu f_2} - \frac{1}{2}\right)$ is the distance from the quantum critical point. The factor in front of the Landau damping term is just the constant $\gamma_0 = 1/\sqrt{2}$. A further discussion of the propagator is given in the main text.

Appendix B

RG Equation at Zero Temperature

To calculate the β functions for r and u one starts from the original action

$$S[n] = \frac{1}{2}\frac{T}{L^d}\sum_q n_q g_q^{-1} n_{-q} + \frac{u_0}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{\substack{k_1,k_2 \\ k_3,k_4}} (n_{k_1} n_{k_2})(n_{k_3} n_{k_4})\, \delta_{k_1+k_2+k_3+k_4}$$

with

$$g^{-1} = \begin{pmatrix} g_3^{-1}\cos^2\phi + g_2^{-1}\sin^2\phi & (g_3^{-1} - g_2^{-1})\sin\phi\cos\phi \\ (g_3^{-1} - g_2^{-1})\sin\phi\cos\phi & g_3^{-1}\sin^2\phi + g_2^{-1}\cos^2\phi \end{pmatrix}$$

One has to divide the action into a low and a high momentum sector, S_l and S_h, as well as a interacting part, S_int. This is done within a loop expansion to first order which means that only such contributions are taken into account that in a diagrammatic representation lead to one-loop contractions only. This corresponds to two high-momentum summations. Within this expansion, the action is split to

$$S_\mathrm{h}[n] = \frac{1}{2}\frac{T}{L^d}\sum_q n_q g_k^{-1} n_{-q}$$

$$S_\mathrm{l}[n] = \frac{1}{2}\frac{T}{L^d}\sum_k n_k g_k^{-1} n_{-k} + \frac{u_0}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{\substack{k_1,k_2 \\ k_3,k_4}} (n_{k_1} n_{k_2})(n_{k_3} n_{k_4})\, \delta_{k_1+k_2+k_3+k_4}$$

$$S_\mathrm{int}[n] = \frac{u_0}{4!}\left(\frac{T}{L^d}\right)^3 \left[2\sum_{\substack{q_1,q_2 \\ k_3,k_4}} (n_{q_1} n_{q_2})(n_{k_3} n_{k_4})\, \delta_{q_1+q_2+k_3+k_4} \right.$$

$$\left. + 4\sum_{\substack{q_1,k_2 \\ q_3,k_4}} (n_{q_1} n_{k_2})(n_{q_3} n_{k_4})\, \delta_{q_1+k_2+q_3+k_4} \right]$$

Here, the convention is such that the label k stands for low momenta and the label q stands for high momenta.

Several terms are neglected due to the one-loop order. First, the quartic term of S_{h} leads to higher order loops and secondly, terms consisting of one slow and three fast momenta (or vice versa) spoil momentum conservation.

Following the usual scheme one calculates the average over the high momentum fields of the exponentiated mixed action, $\left\langle e^{-S_{\mathrm{int}}[n_{\mathrm{h}},n_{\mathrm{l}}]}\right\rangle_{h} = \int \mathcal{D}n_{\mathrm{h}}\, e^{-S_{\mathrm{int}}[n_{\mathrm{h}},n_{\mathrm{l}}]}e^{-S_{\mathrm{h}}[n_{\mathrm{h}}]}$. Within the actual level of accuracy one can expand $e^{-S_{\mathrm{int}}}$ to second order, take the average and after that re-exponentiate the terms, which leads to

$$\left\langle e^{-S_{\mathrm{int}}[n_{\mathrm{h}},n_{\mathrm{l}}]}\right\rangle_{\mathrm{h}} = \exp\left[-\left\langle S_{\mathrm{int}}[n_{\mathrm{h}},n_{\mathrm{l}}]\right\rangle_{\mathrm{h}} + \frac{1}{2}\left\langle S_{\mathrm{int}}[n_{\mathrm{h}},n_{\mathrm{l}}]^{2}\right\rangle_{\mathrm{h}}^{c}\right] \tag{B.1}$$

The superscript c denotes that only connected diagrams are taken into account since disconnected diagrams of this order are either two-loop diagrams or contain powers of n larger than 4. Although such higher power terms will appear, they can be safely neglected since all terms $n^{m>4}$ are irrelevant in a RG sense .

B.1 Mass Correction

The first term in Eq. B.1 yields a renormalization of the mass r_0. The explicit representation of this term is given by

$$\langle S_{\mathrm{int}}[n_{\mathrm{h}},n_{\mathrm{l}}]\rangle_{\mathrm{h}} = 2\frac{u}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{k_1,k_2,k_3} \langle n_{k_1}^i\, n_{k_2}^i\rangle_{\mathrm{h}}\, n_{k_3}^j\, n_{-k_1-k_2-k_3}^j$$
$$+ 4\frac{u}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{k_1,k_2,k_3} n_{k_1}^i \langle n_{k_2}^i\, n_{k_3}^j\rangle_{\mathrm{h}} n_{-k_1-k_2-k_3}^j$$

with an implicit sum over $\{i,j\}$. The propagator $\langle n_k^i n_{k'}^j\rangle = g_{k,k'}^{i,j}$ is given by

$$g = \begin{pmatrix} g_3\cos^2\phi + g_2\sin^2(2\phi)+ & (g_3-g_2)\sin(2\phi)\cos(2\phi) \\ (g_3-g_2)\sin(2\phi)\cos(2\phi) & g_3\sin^2(2\phi)+g_2\cos^2(2\phi) \end{pmatrix}$$

Since the propagator is momentum-diagonal one gets a free summation over the high momenta/frequencies

$$\frac{T}{L^d}\sideset{}{'}\sum_{\mathbf{q},\omega} g^{i,j}(\mathbf{q},\omega) = \int'\frac{\mathrm{d}\omega}{2\pi}\mathrm{d}q\, q \int_0^{2\pi}\frac{\mathrm{d}\phi}{(2\pi)^2}\, g^{i,j}(q,\phi,\omega)$$

The prime over summation and integration respectively denotes, that it runs over the high momentum/frequency-shell depicted in Fig. 4.1 only.

The g_2 mode and g_3 mode do not depend on ϕ, hence, this integration can be

carried out directly and yields

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \, g^{11}(q,\phi,\omega) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \left[g_3 \sin^2\phi + g_2 \cos^2\phi \right] = K_2 \frac{g_3 + g_2}{2}$$

$$= \int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \, g^{22}(q,\phi,\omega)$$

$$\int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \, g^{12}(q,\phi,\omega) = \int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \, g^{21}(q,\phi,\omega)$$

$$= \int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \left[g_3 - g_2 \right] \sin\phi \cos\phi = 0$$

Here, K_2 denotes the averaged volume of the 2-dimensional unit sphere. Performing the summation over $\{i,j\}$ one gets a total mass correction

$$\delta r = \frac{2(N+2)K_2}{4!} u \int' \frac{\mathrm{d}\omega}{2\pi} \mathrm{d}k \, k \left(\frac{1}{r_0 + k^2 + \gamma_0 |\omega|/k} + \frac{1}{r_0 + k^2 + (\omega/k)^2} \right)$$

Here, N is the number of components of the field, which in this problem is $N = 2$. Note the factor of 2 due to the factor $1/2$ in front of the quadratic term of the action.

To obtain the β function, one has to take the derivative with respect to $\ln b$. Neglecting an integration regime of order $(b-1)^2$ one may rewrite the above integration as

$$\delta r = \frac{2(N+2)K_2}{4!} u \int_{\Lambda/b}^{\Lambda} \mathrm{d}k \, k \int_{-\Gamma}^{\Gamma} \frac{\mathrm{d}\omega}{2\pi} \left(\frac{1}{r_0 + k^2 + \gamma_0 |\omega|/k} + \frac{1}{r_0 + k^2 + (\omega/k)^2} \right)$$

$$+ \frac{2(N+2)K_2}{4!} u \int_0^{\Lambda} \mathrm{d}k \, k \left(\int_{\Gamma/b^z}^{\Gamma} + \int_{-\Gamma/b^z}^{-\Gamma} \right) \frac{\mathrm{d}\omega}{2\pi} \left(\frac{1}{r_0 + k^2 + \gamma_0 |\omega|/k} + \frac{1}{r_0 + k^2 + (\omega/k)^2} \right)$$

Then b appears in one of the boundaries only, and the derivation with respect to $\ln b$ is carried out very easily.

$$\frac{\partial(\delta r)}{\partial \ln b} = \frac{K_2 u}{6\pi} \left\{ \frac{\Lambda}{b} \int_{-\Gamma}^{\Gamma} \mathrm{d}\omega \left(\frac{\Lambda}{r_0 + \Lambda^2 + \gamma_0 |\omega|/\Lambda} + \frac{\Lambda}{r_0 + \Lambda^2 + (\omega/\Lambda)^2} \right) \right.$$

$$\left. + 2z\Gamma b^{-z} \int_0^{\Lambda} \mathrm{d}k \, k \left(\frac{k}{r_0 + k^2 + \gamma_0 \Gamma/k} + \frac{k}{r_0 + k^2 + (\Gamma/k)^2} \right) \right\}$$

An expansion to first order in the coupling constant r and setting $b = 1$ yields two terms, f_2 and f_3, originating from the g_2 mode and the g_3 mode, respectively. These terms are explicitly given by

$$f_2 = \frac{K_2 u}{6\pi} \left\{ 2\Lambda^2 \arctan\left(\frac{\Gamma}{\Lambda^2} \right) + \frac{z}{2} \Gamma \ln\left(1 + \frac{\Lambda^4}{\Gamma^2} \right) \right.$$

$$\left. - r_0 \left[\arctan\left(\frac{\Gamma}{\Lambda^2} \right) + \frac{z}{2} \operatorname{arccot}\left(\frac{\Gamma}{\Lambda^2} \right) + \frac{\Lambda^2}{\Gamma} \left(1 - \frac{z}{2} \right) \frac{1}{1 + \frac{\Lambda^4}{\Gamma^2}} \right] \right\}$$

$$f_3 = \frac{K_2 u}{6\pi} \left\{ 2\frac{\Lambda^3}{\gamma} \ln\left(1 + \frac{\gamma\Gamma}{\Lambda^3} \right) + 2\frac{z}{3} \Gamma \ln\left(1 + \frac{\Lambda^3}{\gamma\Gamma} \right) - \frac{z}{3} \frac{r_0 \Gamma}{(\gamma\Gamma)^{2/3}} \left[2(\gamma\Gamma)^{2/3} \frac{\Lambda}{\Lambda^3 + \gamma\Gamma} \left(\frac{3}{z} - 1 \right) \right. \right.$$

$$\left. \left. + \frac{\sqrt{3}\pi}{9} + \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\Lambda - (\gamma\Gamma)^{1/3}}{\sqrt{3}(\gamma\Gamma)^{1/3}} \right) + \frac{1}{3} \ln\left(\frac{\left(\Lambda + (\gamma\Gamma)^{1/3} \right)^2}{(\gamma\Gamma)^{2/3} - \Lambda(\gamma\Gamma)^{1/3} + \Lambda^2} \right) \right] \right\}$$

The main focus is on the asymptotic behavior of the RG flow. Since $r(b)$ flows like b^2 times some small correction, terms independent of r lead to extra contributions for small b but not to the asymptotic flow. The b dependence of u has no strong influence, because for $d + z = 4$ the engineering dimension of u is zero, and the anomalous dimension is small. To describe the asymptotic flow, the contributions for small b can be absorbed into a renormalization of the bare parameter r_0. This renormalized bare mass, r, is now the starting point for the asymptotic flow, $r = r(b = 1)$.

Moreover, the contributions of f_3 for large γ are at least of the order of $\gamma^{-2/3}$ or smaller. Because of the increase of $\gamma(b) = \gamma_0 b$, these terms are asymptotically negligible compared to the γ-independent f_2 contributions. Therefore, the RG flow of r is goverened by the g_2 mode only.

With $z = 2$ one can use the equality $\arctan x + \text{arccot}\, x = \pi/2$ for the f_2 contribution. Adding the engineering dimension of r, one ends up with the resulting asymptotic RG equation

$$\frac{\partial r(b)}{\partial \ln b} = 2r(b) - \frac{u(b)K_2}{12}r(b)$$

B.2   Vertex Correction

The second term in Eq. (B.1) yields a renormalization of the vertex u. This term is given by

$$\frac{1}{2}\langle S_{\text{int}}[n_{\text{h}}, n_{\text{l}}]^2\rangle_{\text{h}}^{\text{c}} = \frac{1}{2}\left(\frac{u_0}{4!}\right)^2\left(\frac{T}{L^d}\right)^4\left[8\sum_{k_1,k_2,k} n_{k_1}^i n_{-k_1+k}^i\left(\sum_q \left\langle n_q^j n_{-q}^k\right\rangle\left\langle n_{q-k}^j n_{k-q}^k\right\rangle\right)n_{k_2}^l n_{-k_2-k}^l\right.$$

$$+ 32\sum_{k_1,k_2,k} n_{k_1}^i\, n_{-k_1+k}^j\left(\sum_q \left\langle n_q^i\, n_{-q}^k\right\rangle\left\langle n_{q-k}^j\, n_{k-q}^k\right\rangle\right)n_{k_2}^l\, n_{-k_2-k}^l$$

$$\left.+ 32\sum_{k_1,k_2,k} n_{k_1}^i\, n_{-k_1+k}^j\left(\sum_q \left\langle n_q^i\, n_{-q}^k\right\rangle\left\langle n_{q-k}^j\, n_{k-q}^l\right\rangle\right)n_{k_2}^k\, n_{-k_2-k}^l\right]$$

with an implicit summation over the indices $\{i, j, k, l\}$. In general, the vertex correction dependends on k. However, assuming that the leading-order correction is k-independent, one may calculate the contribution for $k = 0$ only. The resulting integrals are of the form

$$\int' \frac{d\omega}{2\pi} dq\, q \int_0^{2\pi} \frac{d\phi}{(2\pi)^2}\, g^{i,j}(q, \phi, \omega)g^{k,l}(q, \phi, \omega)$$

where the prime denotes an integral over the high momentum/frequency-shell as before. Due to the fact that the k-dependence is neglected, the angular integration can be carried out easily and leads to the the following results

- Integration over a diagonal term times a non-diagonal term vanishes

- Two non-diagonal terms yield $K_2/8 \, (g_2^2 + g_3^2 - 2g_3g_2)$

- $\int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \, g^{i,i}(q, \phi, \omega) g^{i,i}(q, \phi, \omega) = K_2/8 \, (3g_2^2 + 3g_3^2 + 2g_3g_2)$

- $\int_0^{2\pi} \frac{\mathrm{d}\phi}{(2\pi)^2} \, g^{1,1}(q, \phi, \omega) g^{2,2}(q, \phi, \omega) = K_2/8 \, (g_2^2 + g_3^2 + 6g_3g_2)$

Summing up all possible contractions depending on the set $\{i, j, k, l\}$ one gets the total vertex correction

$$\delta u = \frac{2K_2}{4!} u_0^2 \int' \frac{\mathrm{d}\omega}{2\pi} \mathrm{d}k \, k \left[(N + 7) \left(g_2^2 + g_3^2 \right) + 2g_2g_3 \right]$$

Again, the integration over the high momentum/frequency-shell depicted in Fig 4.1 can be approximated by the integrations over the stripes $[0, \Lambda] \times [\pm\Gamma/b, \pm\Gamma]$ and $[\Lambda/b, \Lambda] \times [\Gamma, \Gamma]$ which leads to the three contributions

$$\delta u = \frac{2K_2}{4!} u_0^2 \int_{\Lambda/b}^{\Lambda} \mathrm{d}k \, k \int_{-\Gamma}^{\Gamma} \frac{\mathrm{d}\omega}{2\pi} \left[(N + 7) \left(g_2^2 + g_3^2 \right) + 2g_2g_3 \right]$$
$$+ \frac{2K_2}{4!} u_0^2 \int_0^{\Lambda} \mathrm{d}k \, k \left(\int_{\Gamma/b^z}^{\Gamma} + \int_{-\Gamma/b^z}^{-\Gamma} \right) \frac{\mathrm{d}\omega}{2\pi} \left[(N + 7) \left(g_2^2 + g_3^2 \right) + 2g_2g_3 \right]$$

To obtain the β function of u, one takes the derivative with respect to $\ln b$ which gives

$$\frac{\partial(\delta u)}{\partial \ln b} = \frac{2K_2}{4!} u_0^2 \left\{ \Lambda b^{-1} \int_{-\Gamma}^{\Gamma} \Lambda \frac{\mathrm{d}\omega}{2\pi} \left[(N + 7) \left(g_2^2 + g_3^2 \right) + 2g_2g_3 \right]_{k=\Lambda} \right.$$
$$\left. + 2z \frac{\Gamma b^{-z}}{2\pi} \int_0^{\Lambda} \mathrm{d}k \, k \left[(N + 7) \left(g_2^2 + g_3^2 \right) + 2g_2g_3 \right]_{\omega=\Gamma} \right\}$$

Splitted into the individual parts coming from the terms g_2^2, g_3^2 and g_2g_3 and expanding to zeroth order in r one gets

$$h_2 = \frac{K_2(N + 7)}{4! \, \pi} u_0^2 \left[\arctan\left(\frac{\Gamma}{\Lambda^2} \right) + \frac{z}{2} \operatorname{arccot}\left(\frac{\Gamma}{\Lambda^2} \right) + \frac{\Lambda^2}{\Gamma} \left(1 - \frac{z}{2} \right) \frac{1}{1 + \frac{\Lambda^4}{\Gamma^2}} \right]$$

$$h_3 = \frac{K_2(N + 7)}{4! \, \pi} u_0^2 \left\{ \frac{2\Lambda}{\Lambda^3 + \gamma\Gamma} \left(1 - \frac{z}{3} \right) - \frac{z}{3} \frac{\Gamma}{(\gamma\Gamma)^{2/3}} \left[\frac{\sqrt{3}\pi}{9} + \frac{2\sqrt{3}}{3} \arctan\left(\frac{2\Lambda - (\gamma\Gamma)^{1/3}}{\sqrt{3}(\gamma\Gamma)^{1/3}} \right) \right. \right.$$
$$\left. \left. + \frac{1}{3} \ln\left(\frac{\left(\Lambda + (\gamma\Gamma)^{1/3} \right)^2}{(\gamma\Gamma)^{2/3} - \Lambda(\gamma\Gamma)^{1/3} + \Lambda^2} \right) \right] \right\}$$

$$h_{\mathrm{mix}} = \frac{2K_2}{4! \, \pi} u_0^2 \left\{ \frac{\Lambda^2}{\gamma^2 + \Lambda^2} \left[2\arctan\left(\frac{\Gamma}{\Lambda^2} \right) + \frac{\gamma}{\Lambda} \ln\left(\frac{\left(\Lambda^3 + \gamma\Gamma \right)^2}{\Lambda^2 \left(\Gamma^2 + \Lambda^4 \right)} \right) \right] \right.$$
$$+ \frac{2z\Lambda}{\gamma} \left[1 + \sqrt{\frac{\Gamma}{8\Lambda}} \left(\arctan\left(1 - \sqrt{\frac{2\Lambda^2}{\Gamma}} \right) - \arctan\left(1 + \sqrt{\frac{2\Lambda^2}{\Gamma}} \right) \right. \right.$$
$$\left. \left. \left. - \frac{1}{2} \ln\left(1 + 2\frac{\sqrt{2\Lambda^2/\Gamma}}{1 + \sqrt{2\Lambda^2/\Gamma} + \Lambda^2/\Gamma} \right) \right) \right] \right\}$$

To obtain the asymptotical leading term one recalls that the constant γ flows according to $\gamma(b) = \gamma_0 b$. Thus, terms containing the inverse of γ are suppressed for large b and their influence on the asymptotic flow can be captured once again in a renormalization of the bare coupling constant u_0. The renormalized bare vertex is the new starting point of the RG flow, $u = u(b = 1)$. This implies that the leading behavior is given by h_2 only, while h_3 and h_{mix} are irrelevant for the asymptotic flow.

With $z = 2$ the resulting RG equation for u reads:

$$\frac{\partial u}{\partial l} = (4 - d - z)u - \frac{3K_2}{16}u^2$$

The term $(4 - d - z)u$ resembles the engineering dimension of the vertex. For two dimensions and a dynamical exponent $z = 2$ it vanishes. Moreover, on has to keep in mind the renormalization of the bare coupling constant $u = u(1) \neq u_0$.

Appendix C

Dimensional Reduction

By using the zero temperature RG flow one obtains an effective low-momentum action with the two cutoffs $T^{1/2}$ and $T^{1/3}$ for the g_2 mode and g_3 mode, respectively. This low-momentum action has the coupling constants r^R and u^R, without the trivial scaling dimension since there is no rescaling to the original cutoff. For the same reason γ_0 is left unchanged. After this cutoff reduction one wants to obtain an effective action for the zero Matsubara mode which is achieved by integrating out all other modes.

One first divide the full action $S[n]$ with the new coupling constants into three parts. The first part for the zero Matsubara mode, $S_0[n_0]$, the second part consists only of non-zero modes, $S[n_m]$ and the third part mixes both modes $S_{\text{mix}}[n_m, n_0]$. The partition function then can be written as

$$\mathcal{Z} = \mathcal{N} \prod_m \int \mathcal{D}n_m \, e^{-S[n]} = \mathcal{N} \int \mathcal{D}n_0 \, e^{-S_0[n_0]} \prod_{m \neq 0} \int \mathcal{D}n_m \, e^{-S[n_m] - S_{\text{mix}}[n_0, n_m]}$$

$$= \int \mathcal{D}n_0 \, e^{-S_{\text{eff}}[n_0]}$$

Conceptually, the step from the first to the second line is similar to the derivation of the effective low-momentum action in the RG scheme in Chap. 4. To one-loop order it corresponds to the evaluation of the diagrams displayed in Fig. 4.2.

Because of the previously applied RG flow, the renormalized bare mass also contains the zero temperature correction for momenta smaller than $T^{1/2}$. Therefore, one has to subtract this contribution, and the resulting effective mass reads

$$R' = r^R + \frac{4(N+2)K_2}{4!} u^R \int_0^\Lambda \mathrm{d}q \, q \left\{ \frac{1}{2\beta} \sum_{\omega_m \neq 0} \left(g_2 \, \Theta(T^{1/2} - q) + g_3 \, \Theta(T^{1/3} - q) \right) \right.$$

$$\left. - \left[\frac{1}{2\beta} \sum_{\omega_m \neq 0} \left(g_2 \, \Theta(T^{1/2} - q) + g_3 \, \Theta(T^{1/3} - q) \right) \right]_{T=0} \right\}$$

The Heaviside step functions resemble the fact of the different cutoffs for the two modes. To perform the Matsubara frequency sum, one applies of the residue the-
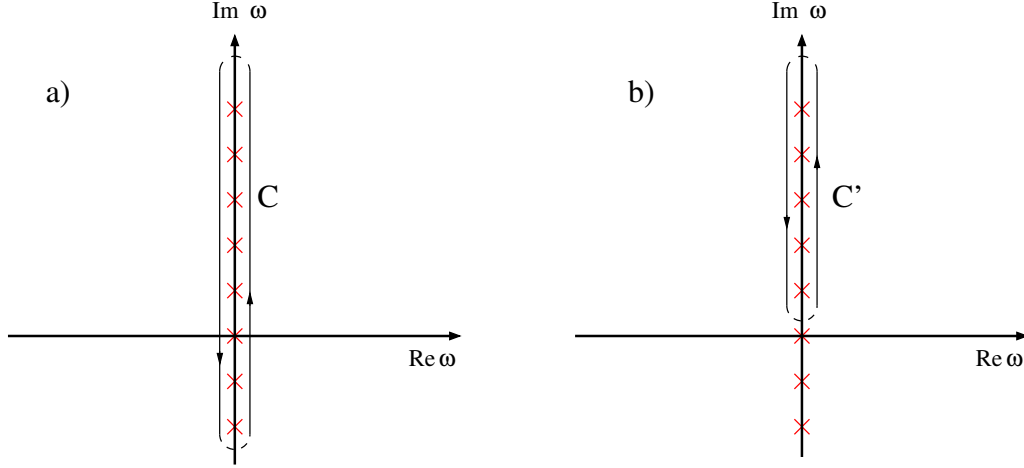
Fig. C.1: The descrete bosonic Matsubara sum $T \sum_m f(\omega_m)$ can be interpreted as a sum over the residues of the function $\coth(\omega/2T) \, f(-i\omega)$ over the contour C. If the sum only extends over positive Matsubara frequencies one has to use the contour C' depicted on the right side and add the zero mode contribution.

orem which for arbitrary non-singular real functions $g(\omega)$ and bosonic Matsubara frequencies reads

$$\sum_{\omega_m} g(\omega_m) = \frac{1}{2T} \int_C \frac{\mathrm{d}z}{2\pi i} \coth\left(\frac{z}{2T}\right) g(-iz)$$

Here, C is the contour depicted in Fig. C.1 (a). To avoid the absolut value in the g_3 mode, one may sum twice over the positive frequencies and add the zero Matsubara mode. For the sum over positive frequencies the residue theorem is applied using the contour C' shown in Fig. C.1 (b). Then one can replace the sum by an integral and obtains

$$\delta r_2 = \frac{2(N+2)K_2}{4!} u^R \int_0^{T^{1/2}} \mathrm{d}q \, q \left\{ \int_C \frac{\mathrm{d}\omega}{2\pi i} \left(\frac{q^2 \left(\coth\left(\frac{\omega}{2T}\right) - 1 \right)}{q^2 (r^R + q^2) - \omega^2} \right) - \frac{2T}{r^R + q^2} \right\}$$

$$\delta r_3 = \frac{2(N+2)K_2}{4!} u^R \int_0^{T^{1/3}} \mathrm{d}q \, q \left\{ \int_C \frac{\mathrm{d}\omega}{2\pi i} \left(\frac{q \left(\coth\left(\frac{\omega}{2T}\right) - 1 \right)}{q(r^R + q^2) - i\gamma_0\omega} \right) - \frac{2T}{r^R + q^2} \right\}$$

Here, δr_2 and δr_3 are the contributions from the g_2 mode and the g_3 mode, respectively. Above the quantum critical point the renormalized mass vanishes, and rescaling according to $q = xT^{1/z}$ and $\omega = yT$ yields

$$\delta r_2 = \frac{(N+2)K_2}{4!} u^R T \int_0^1 \mathrm{d}x \, x^3 \left\{ \int_C \frac{\mathrm{d}y}{2\pi i} \left(\frac{2 \left(\coth y - 1 \right)}{x^4 - y^2} \right) - \frac{2}{x^2} \right\}$$

$$\delta r_3 = \frac{(N+2)K_2}{4!} u^R T \int_0^1 \mathrm{d}x \, x^2 \left\{ \int_C \frac{\mathrm{d}y}{2\pi i} \left(\frac{2 \left(\coth y - 1 \right)}{q^3 - i\gamma_0 y} \right) - \frac{2}{x^2} \right\}$$

Both integrals are constant, and one finds

$$R' = u^R \kappa \, T = 2c\kappa \frac{T}{\ln(c_1/T)}$$

82

with $c_1 = e^{2c/u}\Lambda^2$ and some unspecified constant κ

The effective vertex of the zero mode, U, can be approximated by the zero temperature renormalized vertex u^R. Since this renormalized vertex is finite, any higher order temperature dependence is suppressed by a factor of T.

Thus, one has obtained an effective action of the zero Matsubara mode with an effective mass R', but it still contains the two different cutoffs for the g_2 mode and the g_3 mode. To get rid of this cutoff structure, one integrates out the intermediate regime $T^{1/3} > k > T^{1/2}$ which leads to another correction to the effective mass. In this intermediate regime only the propagator of the mode with $z = 3$ is non-zero, and thus one gets the following correction

$$R = R' + \frac{4(N+2)K_2}{4!}U \int_{T^{1/2}}^{\Lambda} \mathrm{d}q \, \frac{q}{2\beta} \left(g_2 \, \Theta(T^{1/2} - q) + g_3 \, \Theta(T^{1/3} - q) \right)$$

$$= R' + \frac{2(N+2)K_2}{4!}u^R \int_{T^{1/2}}^{T^{1/3}} \mathrm{d}q \, q \frac{T}{R' + q^2}$$

The integral can be easily carried out yielding $\frac{T}{2}\ln\left(\frac{R'+T^{2/3}}{R'+T}\right)$. Substituting the obtained formula for R' one arrives at the expression

$$R \sim \frac{T}{\ln(c_1/T)} + \frac{\tilde{c}T}{\ln(1/T)} \ln\left(1 + \frac{T^{2/3}\ln(1/T) - T\ln(1/T)}{aT + T\ln(1/T)} \right)$$

$$\approx \frac{T}{\ln(c_1/T)} + \frac{\tilde{c}\, T}{\ln(1/T)} \frac{\ln(1/T)}{3} \sim T$$

with some constants $\tilde{c}$ and a. To leading order, the effective mass of the zero mode action is, therefore, linear in temperature. A further discussion of this result is given in the main text.

Appendix D

RG Equation at Finite Temperatures

At finite temperatures one has to calculate the RG equation for the mass r only, while the vertex u is to leading order governed by its zero-temperature behavior. The one-loop correction to the low-momentum action reads

$$\langle S_{\text{int}}[n_{\text{h}}, n_{\text{l}}]\rangle_{\text{h}} = 2\frac{u}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{k_1,k_2,k_3} \left\langle n^i_{k_1}\, n^i_{k_2}\right\rangle_{\text{h}} n^j_{k_3}\, n^j_{-k_1-k_2-k_3}$$

$$+ 4\frac{u}{4!}\left(\frac{T}{L^d}\right)^3 \sum_{k_1,k_2,k_3} n^i_{k_1} \left\langle n^i_{k_2} n^j_{k_3}\right\rangle_{\text{h}} n^j_{-k_1-k_2-k_3}$$

$$= \frac{2(N+2)u}{4!}\left(\frac{T}{L^d}\right)^2 \sum_{k,q} \frac{g_2(q) + g_3(q)}{2}\Bigg|_{\text{h}} n^j_k\, n^j_{-k}$$

where the sum over $k = (\mathbf{k}, \omega_n)$ runs over discrete Matsubara frequencies. In the second line, it was used that $\left\langle n^i_{k_1}\, n^j_{k_2}\right\rangle \propto \delta_{k_1,-k_2}$ and $\sum_k \left\langle n^i_q\, n^j_{-q}\right\rangle = \delta_{i,j} \sum_q \frac{g_2(q)+g_3(q)}{2}$, see App. B.

For the g_2 mode one has to calculate

$$\frac{T}{L^d} \sum_{\mathbf{q},\omega_n} g_2(\mathbf{q},\omega_n) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} T \sum_{\omega_n} \frac{q^2}{q^2(r+q^2) + \omega_n^2}$$

$$= \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{2} \int_C \frac{\mathrm{d}\omega}{2\pi i} \coth\left(\frac{\omega}{2T}\right) \frac{q^2}{q^2(r+q^2) - \omega^2}$$

In the second line, the residue theorem is applied to switch from the Marsubara summation to an integration over the counter C shownin Fig. D.1 (a).

Deforming the contour C into the contour C', depicted in Fig. D.1 (b), one apply the residue theorem, replacing the above integral by the sum of the residues at the
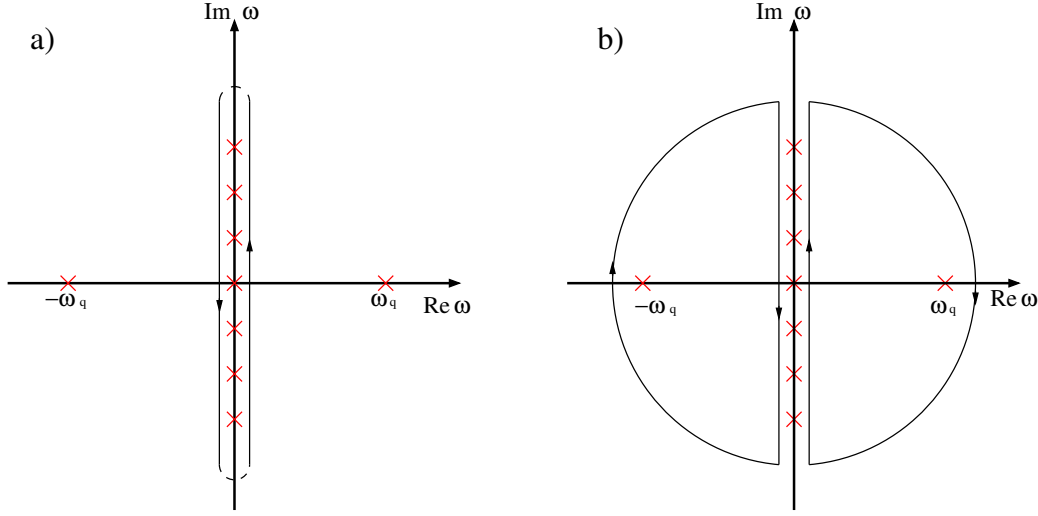
Fig. D.1: Illustration of the contour integration for the Matsubara frequencies. In (a) contour C is depicted. This contour is then deformed into the contour C' depicted in (b).

two simple poles $\pm\omega_q = \pm q\sqrt{r+q^2}$ which leads to

$$-\frac{1}{2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\left[\mathrm{Res}\left(\frac{q^2\coth\left(\omega/(2T)\right)}{\omega_q^2-\omega^2},\omega_q\right)+\mathrm{Res}\left(\frac{q^2\coth\left(\omega/(2T)\right)}{\omega_q^2-\omega^2},-\omega_q\right)\right]$$

$$=\frac{1}{2}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\coth\left(\frac{\omega_q}{2T}\right)\frac{q^2}{\omega_q}$$

The minus sign in the first line arises from the fact that the contour C' is mathematically negatively orientated.

Carrying out the frequency integral one has to be a bit more specific about the cutoff. The calculation is performed using a hard cutoff scheme on the real ω-axis. This means that the original summation over large ω_n is done such that after analytical continuation to the real axis the integration is sharply cut off at values $\pm\Gamma$. In particular, one chooses a cutoff Γ which is larger than the pole ω_q for all momenta $q < \Lambda$.

To obtain the β function for the mass r, one restricts the integrals to the high momentum/frequency-shell, as it is done in the zero temperature case. This integration can be approximated, as before, by the integration over three stripes, namely $[\Lambda/b,\Lambda]\times[-\Gamma,\Gamma]$ and $[0,\Lambda]\times[\pm\Gamma/b,\pm\Gamma]$. Differentiating with respect to $\ln b$ leads for the g_2 mode to

$$f_2(r,\gamma_0,T) = K_2\frac{\partial}{\partial\ln b}\int_{\Lambda/b}^{\Lambda}\mathrm{d}q\,q\coth\left(\frac{q\sqrt{r+q^2}}{2T}\right)\frac{q}{2\sqrt{r+q^2}}$$

$$= K_2\frac{\Lambda^3}{2\sqrt{r+\Lambda^2}}\coth\left(\frac{\Lambda\sqrt{r+\Lambda^2}}{2T}\right)$$

To eliminate the absolute value of ω_n for the g_3 mode, one may sum twice over
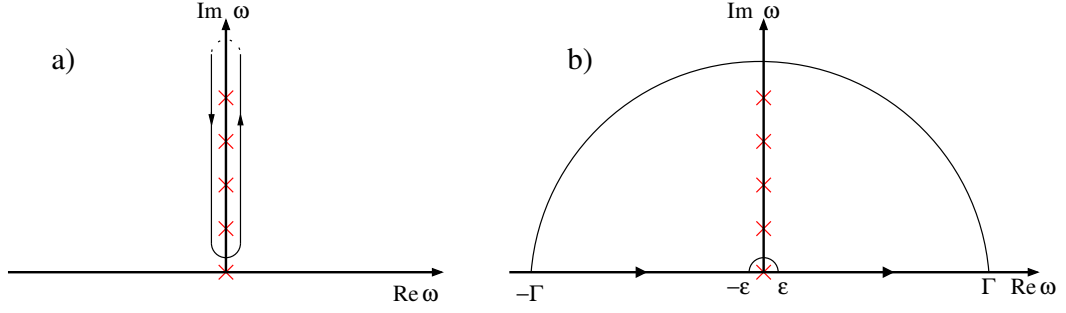
Fig. D.2: On the left side the contour S is depicted. This contour is deformed into the contour S' depicted on the right side.

positive Matsubara frequencies while adding the zero mode contribution:

$$\frac{T}{L^d} \sum_{\mathbf{q},\omega_n} g_3(\mathbf{q},\omega_n) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} T \sum_{\omega_n} \frac{q}{q(r+q^2) + \gamma_0 |\omega_n|}$$

$$= \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \left[\frac{T}{r+q^2} + 2T \sum_{\omega_n > 0} \frac{q}{q(r+q^2) + \gamma_0 \omega_n} \right]$$

As before one applies the residue theorem to express the Matsubara frequency sum by an integral over the contour S displayed in Fig. D.2 (a). This contour is deformed into the contour S' shown in Fig. D.2 (b). The term in the brackets then becomes

$$\frac{T}{r+q^2} + \int_{S'} \frac{\mathrm{d}\omega}{2\pi i} \coth\left(\frac{\omega}{2T}\right) \frac{q}{q(r+q^2) - i\gamma_0 \omega}$$

The integral over the contour S' consists of three parts: the one over the large semi-circle from $-\Gamma$ to Γ, the one over the infinitesimal semi-circle from $-\epsilon$ to ϵ and the principal value integral from $-\Gamma$ to Γ. While the large semi-circle is negligible, the small one gives

$$-\int_{S_\epsilon(0)} \frac{\mathrm{d}\omega}{2\pi i} \coth \frac{\omega}{2T} \frac{q}{q(r+q^2) - i\gamma_0 \omega} = -\int_0^\pi \frac{\mathrm{d}\phi}{2\pi i} \coth \frac{\epsilon e^{i\phi}}{2T} \frac{iq\epsilon e^{i\phi}}{q(r+q^2) - i\gamma_0 \epsilon e^{i\phi}}$$

$$\simeq -\frac{Tq}{\pi} \int_0^\pi \frac{\mathrm{d}\phi}{q(r+q^2) - i\gamma_0 \epsilon e^{i\phi}} = \frac{2T}{\pi} \frac{\arctan\left(\frac{\epsilon}{q(q^2+r)}\right) - \frac{\pi}{2}}{r+q^2}$$

The minus sign arises from the orientation, and the approximation $\coth x \simeq 1/x$ for small $|x|$ was done. In the limit $\epsilon \to 0^+$, the result simplifies to $\frac{-T}{r+q^2}$, and, therefore, cancels against the zero mode contribution. The g_3 part then yields

$$\frac{T}{L^d} \sum_{\mathbf{k},\omega_n} g_3(\mathbf{k},\omega_n) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \mathcal{P} \int_{-\Gamma}^{\Gamma} \frac{\mathrm{d}\omega}{2\pi i} \coth\left(\frac{\omega}{2T}\right) \frac{q}{q(r+q^2) - i\gamma_0 \omega}$$

$$= \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \int_0^{\Gamma} \frac{\mathrm{d}\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{q\gamma_0 \omega}{q^2(r+q^2)^2 + \gamma_0^2 \omega^2}$$

87

Restricting the integration to the stripes $[\Gamma/b, \Gamma] \times [0, \Lambda]$ and $[0, \Gamma] \times [\Lambda/b, \Lambda]$, one takes the derivative with respect to $\ln b$ and gets the two contributions

$$f_3(r, \gamma_0, T) = K_2 \frac{\partial}{\partial \ln b} \left[\int_{\Lambda/b}^{\Lambda} dq\, q \int_0^{\Gamma} \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{q\gamma_0\omega}{q^2(r + q^2)^2 + \gamma_0^2\omega^2} \right.$$
$$\left. + \int_0^{\Lambda} dq\, q \int_{\Gamma/b^z}^{\Gamma} \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{q\gamma_0\omega}{q^2(r + q^2)^2 + \gamma_0^2\omega^2} \right]$$

$$= K_2\Lambda^3 \int_0^{\Gamma} \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{\gamma_0\omega}{\Lambda^2(r + q^2)^2 + \gamma_0^2\omega^2}$$
$$+ \frac{z\Gamma^2}{\pi} \int_0^{\Lambda} dq \coth\left(\frac{\Gamma}{2T}\right) \frac{q^2\gamma_0}{q^2(r + q^2)^2 + \gamma_0^2\Gamma^2}$$

To cast the RG equation in a more convenient form, one can absorb the engineering dimension by switching to the variable $R = rb^{-2}$. The β function then takes the form

$$\frac{\partial R}{\partial \ln b} = \frac{2(N + 2)}{4!} u(b)b^{-2} \left[f_2(Rb^2, \gamma(b), T(b)) + f_3(Rb^2, \gamma(b), T(b)) \right]$$

As in the zero temperature case, γ flows according to $\gamma(b) = \gamma_0 b$. Since temperature is related to the Matsubara frequencies, $\omega_n = 2\pi nT$, it has to obey the same RG equation than frequencies, namely $T(b) = Tb^z$.

To extract the temperature and momentum dependence of R, the RG equation may be integrated over $d(\ln b) = db/b$, yielding

$$R(b') - R(1) = \frac{(N + 2)K_2}{4!} \int_1^{b'} \frac{db}{b} u(b) \left[\frac{b^{-3}\Lambda^3}{\sqrt{R(b) + b^{-2}\Lambda^2}} \coth\left(\frac{\Lambda\sqrt{R(b) + b^{-2}\Lambda^2}}{2Tb}\right) \right.$$
$$+ 2b^{-2}\Lambda^3 \int_0^{\Gamma} \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2Tb^2}\right) \frac{b\gamma_0\omega}{b^4[\Lambda^2(R(b) + b^{-2}\Lambda^2)^2 + \gamma_0^2 b^{-2}\omega^2]}$$
$$\left. + b^{-2} \frac{4\Gamma^2}{\pi} \int_0^{\Lambda} dq \coth\left(\frac{\Gamma}{2Tb^2}\right) \frac{q^2 b\gamma_0}{b^4[q^2(R(b) + b^{-2}q^2)^2 + \gamma_0^2 b^{-2}\Gamma^2]} \right]$$

By substituting $b = \Lambda/k$ and rescaling $\omega \to \omega k^2/\Lambda^2$ and $q \to q\Lambda/k$ one finally gets

$$R(b) - R(1) = \int_{\Lambda/b}^{\Lambda} dk\, F(k, T, R(\Lambda/k))$$
$$= \frac{(N + 2)K_2}{4!} \int_{\Lambda/b}^{\Lambda} dk\, u(\Lambda/k) \left[\frac{k^2}{\sqrt{R(\Lambda/k) + k^2}} \coth\left(\frac{k\sqrt{R(\Lambda/k) + k^2}}{2T}\right) \right.$$
$$+ 2k^2 \int_0^{\Gamma'} \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{\gamma_0\omega}{k^2(R(\Lambda/k) + k^2)^2 + \gamma_0^2\omega^2}$$
$$\left. + \frac{4\Gamma'^2}{\pi} \int_0^{k} dq \coth\left(\frac{\Gamma'}{2T}\right) \frac{q^2\gamma_0}{q^2(R(\Lambda/k) + q^2)^2 + \gamma_0^2\Gamma'^2} \right] \quad \text{(D.1)}$$

where Γ' is the rescaled cutoff $\Gamma\Lambda^2/k^2$.

In the quantum critical regime, the zero temperature renormalization of the bare mass $R(1)$ vanishes so that

$$R(b) = \underbrace{R(1) + \int_{\Lambda/b}^{\Lambda} dk\, F(k,0,R(\Lambda/k))}_{=0} + \int_{\Lambda/b}^{\Lambda} dk\, [F(k,T,R(\Lambda/k)) - F(k,0,R(\Lambda/k))]$$

The subtraction of the zero temperature renormalization results only in substituting $\coth x \to \coth x - 1$.

To find the leading-order behavior in which one is interested, the different terms have to be compared. First of all, the last term in Eq. (D.1) can be neglected due to the factor $\coth\left(\frac{\Gamma'}{2T}\right) - 1 \approx 2\exp\left(-\frac{\Gamma k^2}{2\Lambda^2 T}\right)$, which is exponentially suppressed for $k^2/T \gg 1$. The remaining frequency integration can be solved by the approximation

$$\coth(x) - 1 = \begin{cases} 0 & \text{for}\, |x| > 1 \\ 1/x - 1 & \text{for}\, |x| < 1 \end{cases}$$

With this simplification one gets

$$2k^2 \int_0^{\Gamma'} \frac{d\omega}{\pi} \coth\left(\frac{\omega}{2T}\right) \frac{\gamma_0 \omega}{k^2(R+k^2)^2 + \gamma_0^2 \omega^2} = 2k^2 \int_0^{2T} \frac{d\omega}{\pi} \frac{2T\gamma_0 - \gamma_0 \omega}{k^2(R+k^2)^2 + \gamma_0^2 \omega^2}$$

$$= \frac{2Tk}{R+k^2} \frac{2}{\pi} \arctan\left(\frac{2T\gamma_0}{k(R+k^2)}\right) + \frac{k^2}{\pi\gamma_0} \ln\left[1 + \left(\frac{2T\gamma_0}{k(R+k^2)}\right)^2\right]$$

Substituting the zero temperature result $u(b) = c/\ln(be^{c/u})$ and using the approximation for $\coth x - 1$ again the result for R reads

$$R\left(\frac{\Lambda}{q}\right) = \frac{8}{9} \int_q^{\Lambda} dk \frac{2T}{k} \frac{1}{\ln(e^{c/u}\Lambda/k)} \left\{ \frac{k^2}{R(\Lambda/k)+k^2} \Theta\left(\frac{2T}{k\sqrt{R(\Lambda/k)+k^2}} - 1\right)\right.$$

$$+ \frac{k^2}{2T} \frac{k}{\sqrt{R(\Lambda/k)+k^2}} \Theta\left(\frac{2T}{k\sqrt{R(\Lambda/k)+k^2}} - 1\right)$$

$$+ \frac{k^2}{R(\Lambda/k)+k^2} \frac{2}{\pi} \arctan\left(\frac{2T\gamma_0}{k(R(\Lambda/k)+k^2)}\right)$$

$$\left. + \frac{1}{\pi} \frac{k^3}{2T\gamma_0} \ln\left[1 + \left(\frac{2T\gamma_0}{k(R(\Lambda/k)+k^2)}\right)^2\right] \right\}$$

In the following, the assumption is made that $R \sim T$, up to logarithmic corrections, implying that $k < (2T)^{1/2}$ for the first two terms due to the Heaviside step function. This restriction implies that one can neglect the second term compared to the first one due to the factor of $k^2/T \ll 1$. Furthermore, the fourth term is negligible compared to the third term for $k < (2\gamma_0 T)^{1/3}$ while they are of same order otherwise. In the end, one may approximate

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x \approx \begin{cases} \frac{\pi}{2} - x & \text{for}\, x \ll 1 \\ 0 & \text{for}\, x \gg 1 \end{cases}$$

Finally, one ends up with

$$
R\left(\frac{\Lambda}{q}\right) = \frac{16\,T}{9} \int_q^\Lambda \frac{dk}{k}\, \frac{1}{\ln(e^{c/u}\Lambda/k)} \left\{ \frac{k^2}{R(\Lambda/k) + k^2}\, \Theta\left(\frac{2T}{k\sqrt{R(\Lambda/k) + k^2}} - 1\right)\right.
$$
$$
\left. + \frac{k^2}{R(\Lambda/k) + k^2}\, \Theta\left(\frac{2\gamma_0 T}{k(R(\Lambda/k) + k^2)} - 1\right)\right\}
$$

The correlation length is given by $\xi = R^{-1/2}(b_\star)$ where $b_\star$ is the point at which the RG flow has to be stopped, i.e. where $r(b_\star) = b_\star^2 R(b_\star) \sim \Lambda^2$. Assuming $R(b)$ to be only weakly depending on b in comparison to the factor b^2, one can may approximate R to be constant. Then $b_\star \Lambda/\sqrt{R}$ which corresponds to $q = \sqrt{R}$. Thus, the self-consistent equation arises

$$
\xi^{-2} = R = \frac{16}{9}T \left[\int_{R^{1/2}}^{T^{1/2}} \frac{dk}{k}\, \frac{1}{\ln(c_1/k)}\, \frac{k^2}{R + k^2} + \int_{R^{1/2}}^{T^{1/3}} \frac{dk}{k}\, \frac{1}{\ln(c_1/k)}\, \frac{k^2}{R + k^2} \right]
$$
$$
\approx \frac{16}{9}T \left[\int_{R^{1/2}}^{T^{1/2}} \frac{dk}{k}\, \frac{1}{\ln(c_1/k)} + \int_{R^{1/2}}^{T^{1/3}} \frac{dk}{k}\, \frac{1}{\ln(c_1/k)} \right]
$$

where the abbreviation $c_1 = e^{c/u}\Lambda$ was introduced, and the approximation $k^2/(R + k^2) \approx 1$ for $k^2 > R$ was made. The integrals can be carried out easily and yield

$$
\xi^{-2} = -\frac{16}{9}T \left[\ln\left(\frac{\ln\left(c_1/T^{1/2}\right)}{\ln\left(c_1/R^{1/2}\right)}\right) + \ln\left(\frac{\ln\left(c_1/T^{1/3}\right)}{\ln\left(c_1/R^{1/2}\right)}\right) \right]
$$

Substituting $R \sim T$, one finally gets

$$
\xi^{-2} = \frac{16}{9}T \ln\left(\frac{\ln(\tilde c_1/T^{1/2})\ln(\tilde c_1/T^{1/2})}{\ln(c_1/T^{1/3})\ln(c_1/T^{1/2})}\right) \approx \frac{16}{9}T \ln\left(\frac{3\ln(1/T) + 4\ln(\tilde c_1)}{2\ln(1/T) + 5\ln(c_1)}\right)
$$
$$
= \frac{16}{9}T \left[\ln\frac{3}{2} - \frac{\text{const.}}{\ln(1/T)} + \mathcal{O}\left(\frac{1}{\ln^2(1/T)}\right) \right]
$$

where $\tilde c_1$ contains c_1 as well as the constant R/T.

Deutsche Zusammenfassung

In dieser Arbeit wird die quadrupolare Pomeranchuck-Instabilität einer isotropen Fermi-Flüssigkeit in zwei Dimensionen untersucht. Die effektive Ginzburg-Landau Theorie für ein System spinloser Fermionen mit quadrupolaren Wechselwirkungen wurde von Oganesyan et al. [1] bereits 2001 hergeleitet.

Die Besonderheit dieser Theorie ist das Auftreten zweier Moden, die gleichzeitig kritisch werden, aber unterschiedliche Dynamiken haben. Zum einen gibt es eine ballistische Mode, deren kritischer dynamischer Exponent $z = 2$ ist, zum anderen taucht eine Laudau-gedämpfte Mode auf mit einen dynamischen Exponenten von $z = 3$. Das Zusammenspiel dieser beiden Moden führt zu interessanten Konsequenzen.

Am Temperatur-Nullpunkt hat die ballistische Mode eine effektive Dimension von $d + z = 4$ und ist somit an der oberen kritischen Dimension. Dies führt bei einer störungstheoretischen Behandlung in jeder Ordnung zu logarithmischen Singularitäten. Auf der anderen Seite wird bei endlichen Temperaturen die Mode mit dem höheren dynamischen Exponenten dominieren, da ihr ein größerer Phasenraum zur Verfügung steht.

Will man die effektive Theorie am absoluten Temperatur-Nullpunkt untersuchen, muss man zunächst die angesprochenen Divergenzen resummieren. Dies wird im Allgemeinen mit Hilfe der Renormierungsgruppen (RG) - Methode gemacht. Allerdings ist diese in ihrer herkömmlichen Form nicht anwendbar, da eine Frequenzreskalierung mit dem dynamischen Exponenten durchgeführt werden muss. Man muss somit zunächst die RG-Methode an dieses spezielle Problem anpassen. Mit diesem angepassten RG-Schema kann dann die Korrelationslänge am Temperatur-Nullpunkt berechnet werden. Man erhält eine Universalitätsklasse, die nicht die des Ising- oder XY-Models ist.

Bei endlichen Temperaturen findet ein äußerst komplexes Zusammenspiel zwischen den beiden Moden statt. Nahe eines Phasenüberganges existiert eine Längenskala $\xi_T \sim T^{-1/z}$, die den sogenannten klassischen Bereich vom Quantenbereich trennt. Für kleinere Längen herrscht der Quanten-Charakter vor, der mit einer effektiven Dimension von $d + z$ einhergeht. Auf größeren Längen kann die Theorie vollständig klassisch in d Dimensionen beschrieben werden. Da es bei diesem Problem jedoch zwei verschiedene dynamische Exponenten gibt, existiert ein Zwischenregime, in dem

die ballistische Mode bereits klassisch ist, während die gedämpfte Mode noch durch den Quantencharakter bestimmt wird. Im dieser Diplomarbeit wird dargelegt, dass genau dieses Zwischenregime die Eigenschaften des Systems bei endlichen Temperaturen dominiert. Dies führt dazu, dass die Theorie der dimensionalen Reduktion zusammenbricht, d.h. das System kann nahe des Phasenüberganges nicht durch eine rein klassische, d-dimensionale Theorie beschrieben werden. Es führt zudem auf eine universelle Temperaturabhängigkeit der Korrelationslänge bei endlichen Temperaturen.

Im letzten Teil werden mit der spezifischen Wärme, der thermischen Ausdehnung und der Kompressibilität die thermodynamischen Eigenschaften des Systems auf Gaußschem Niveau untersucht. Wie erwartet, werden die spezifische Wärme und die thermische Ausdehnung durch die gedämpfte Mode dominiert. Die Kompressibilität wird von der ballistischen Mode dominiert, zeigt jedoch Divergenzen, die das Einbeziehen von Wechselwirkungen notwendig machen.

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Erklärung

Hiermit versichere ich, die vorliegende Arbeit selbstständig und ohne fremde Hilfe angefertigt zu haben. Verwendete Literatur und andere Unterlagen sind jeweils im Text vermerkt und im Anhang aufgelistet.

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Mario Zacharias

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