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Advanced Quantum Mechanics

Exercise sheet 2

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **October 20th**, 2014 (10 am, i.e. before the lecture starts)

5. Fock-space (5 points)

A four dimensional single-particle Hilbert space is given by the four states: $\{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle\}$. A general normalized wavefunction in Fock-space is denoted by $|n_1, n_2, n_3, n_4\rangle$ where n_i with $i = 1, 2, 3, 4$ denote the number of particles occupying state $|\Psi_i\rangle$.

a) What is the dimension of the Hilbert space for N bosons?
Hint: $\sum_{n=0}^{N} n = N(N+1)/2$ and $\sum_{n=0}^{N} n^2 = N(N+1)(2N+1)/6$.

b) What is the dimension of the Hilbert space for $N = 0, 1, 2, 3, 4$ fermions, respectively?

c) Consider the bosonic wavefunction $|1, 2, 1, 0\rangle$ in Fock representation. Write this wavefunction explicitly in terms of the single-particle eigenfunctions $|\Psi_i\rangle$ and also in real-space using $\Psi_i(x) = \langle x|\Psi_i\rangle$.

d) Consider the fermionic wavefunction $|1, 1, 0, 1\rangle$ in Fock representation. Write this wavefunction explicitly in terms of single-particle ket-wavefunctions as well as in the real-space representation.

6. Two spinful fermions (7 points)

Two fermionic annihilation operators, $\mathbf{f}_\sigma$, with $\sigma =\uparrow, \downarrow$, and the corresponding creation operators, $\mathbf{f}_\sigma^\dagger$, are given, i.e., $\{\mathbf{f}_\mu, \mathbf{f}_\nu\} = 0$, $\{\mathbf{f}_\mu^\dagger, \mathbf{f}_\nu^\dagger\} = 0$ and $\{\mathbf{f}_\mu^\dagger, \mathbf{f}_\nu\} = \delta_{\mu\nu}$. Consider the total number operator $\mathbf{n} = \mathbf{n}_\uparrow + \mathbf{n}_\downarrow$, with $\mathbf{n}_\sigma = \mathbf{f}_\sigma^\dagger \mathbf{f}_\sigma$, and the vector operator $\vec{\mathbf{S}} = \frac{\hbar}{2} \mathbf{f}_\mu^\dagger \vec{\sigma}_{\mu\nu} \mathbf{f}_\nu$ where $\vec{\sigma}$ is the vector of Pauli matrices. The wavefunction in Fock-space is denoted as $|n_\uparrow, n_\downarrow\rangle$.

a) What is the dimension of this Fock space? Compute $\mathbf{n}|n_\uparrow, n_\downarrow\rangle$ for all states.

b) Show that $\vec{\mathbf{S}}$ obeys the angular momentum algebra, i.e., $[\mathbf{S}^i, \mathbf{S}^j] = i\hbar\epsilon_{ijk}\mathbf{S}^k$.

c) Show that the operators, $\mathbf{n}$ and $\vec{\mathbf{S}}$, commute, i.e., $[\mathbf{n}, \vec{\mathbf{S}}^i] = 0$ for all $i = 1, 2, 3$.

d) As the operator $\vec{\mathbf{S}}$ commutes with $\mathbf{n}$ we can consider its properties within the subspaces with total number of particles, $N = 0, 1$ and 2, separately. Show that within the subspaces with $N = 0$ and $N = 2$, the operator $\vec{\mathbf{S}}$ reduces to zero, i.e., $\vec{\mathbf{S}}|0, 0\rangle = 0$ and $\vec{\mathbf{S}}|1, 1\rangle = 0$. Show that within the $N = 1$ subspace, the operator $\vec{\mathbf{S}}$ corresponds to a spin-$\frac{1}{2}$, i.e., $\vec{\mathbf{S}}^2|\Psi\rangle = \hbar^2 3/4|\Psi\rangle$ with $|\Psi\rangle = \alpha|1, 0\rangle + \beta|0, 1\rangle$.

7. Two particles (8 points)

Consider creation and annihilation operators in real space, $\mathbf{\Psi}^\dagger(x)$ and $\mathbf{\Psi}(x)$, respectively. A specific Fock state containing two particles is given by

$$|1,1\rangle = \int dxdy\varphi_1(x)\varphi_2(y)\mathbf{\Psi}^\dagger(x)\mathbf{\Psi}^\dagger(y)|\text{vac}\rangle,$$

where $\varphi_i(x)$ are orthonormal wavefunctions for states labeled by the quantum number i. The state $|\text{vac}\rangle$ is the normalized vacuum state that contains no particles, i.e., $\mathbf{\Psi}(x)|\text{vac}\rangle = 0$ for all x. In the case of bosons the operators satisfy $[\mathbf{\Psi}(x), \mathbf{\Psi}^\dagger(y)] = \delta(x-y)$ and $[\mathbf{\Psi}(x), \mathbf{\Psi}(y)] = [\mathbf{\Psi}^\dagger(x), \mathbf{\Psi}^\dagger(y)] = 0$. In the case of fermions we have instead $\{\mathbf{\Psi}(x), \mathbf{\Psi}^\dagger(y)\} = \delta(x-y)$ and $\{\mathbf{\Psi}(x), \mathbf{\Psi}(y)\} = \{\mathbf{\Psi}^\dagger(x), \mathbf{\Psi}^\dagger(y)\} = 0$. It is convenient to combine the two cases using $\mathbf{\Psi}(x)\mathbf{\Psi}^\dagger(y) = \delta(x-y) + \sigma\mathbf{\Psi}^\dagger(y)\mathbf{\Psi}(x)$ and $\mathbf{\Psi}(x)\mathbf{\Psi}(y) = \sigma\mathbf{\Psi}(y)\mathbf{\Psi}(x)$ and $\mathbf{\Psi}^\dagger(x)\mathbf{\Psi}^\dagger(y) = \sigma\mathbf{\Psi}^\dagger(y)\mathbf{\Psi}^\dagger(x)$ with $\sigma = 1$ for bosons and $\sigma = -1$ for fermions.

a) Evaluate $\mathbf{\Psi}(x)\mathbf{\Psi}(y)|1,1\rangle$ for bosons and fermions by first commuting the annihilation operators with the creation operators and then applying them to the vacuum.

b) Consider the wavefunction $|x', y'\rangle = \mathbf{\Psi}^\dagger(x')\mathbf{\Psi}^\dagger(y')|\text{vac}\rangle$. Evaluate the overlap $\langle x', y'|1,1\rangle$ for the case of bosons and fermions using the result of part (a).

c) Evaluate the expectation value for the density $\langle 1,1|\mathbf{\Psi}^\dagger(x)\mathbf{\Psi}(x)|1,1\rangle$ for bosons and fermions.