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Advanced Quantum Mechanics

Exercise sheet 4

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **November 3rd**, 2014 (10 am, i.e. before the lecture starts)

10. Schwinger boson representation (5 points)

The Schwinger boson provides a representation of quantum mechanical spins in terms of bosons. The spin is written in terms of two bosonic operators a and b in the form

$$\hat{S}^+ = a^\dagger b,\ \hat{S}^- = \left(\hat{S}^+\right)^\dagger$$
$$\hat{S}_z = \frac{1}{2}(a^\dagger a - b^\dagger b)\,.$$

a) Show that this definition is consistent with the commutation relations for the spin operator.

b) Derive the constraint on the bosonic Hilbert space that comes from requiring a fixed spin quantum number S.

c) Show that

$$|S,m\rangle = \frac{\left(a^\dagger\right)^{S+m}}{\sqrt{(S+m)!}} \frac{\left(b^\dagger\right)^{S-m}}{\sqrt{(S-m)!}} |\Omega\rangle,$$

with Ω being the vacuum state of the Schwinger bosons, is an eigenstate of $\mathbf{S}^2$ and S_z.

11. Bose condensate wavefunction (5 points)

a) The ground state of a Bose condensate $|\psi_0\rangle$ is defined by the property $\tilde{\mathbf{c}}_\mathbf{k}|\psi_0\rangle = 0$, where $\tilde{\mathbf{c}}_\mathbf{k} = \mathbf{c}_\mathbf{k} - \frac{\alpha}{\mu}\delta_{\mathbf{k},0}$ and $\tilde{\mathbf{c}}^\dagger_\mathbf{k} = \mathbf{c}^\dagger_\mathbf{k} - \frac{\alpha^*}{\mu}\delta_{\mathbf{k},0}$ are shifted bosonic operators. Determine the normalized ground state wave function.
Hint: Use the ansatz $|\psi_0\rangle = \sum_{n=0}^{\infty} a_n(\mathbf{c}^\dagger_0)^n|0\rangle$ and determine the coefficients a_n.

12. Fermionic Bogoliubov transformation (10 points)

a) Consider fermionic creation and annihilation operators, $\mathbf{c}^\dagger_{\mathbf{k}\sigma}$ and $\mathbf{c}_{\mathbf{k}\sigma}$, respectively, where $\mathbf{k}$ labels the momentum and $\sigma = \uparrow, \downarrow$. New operators are introduced with the help of the transformation

$$\mathbf{d}_{\mathbf{k}\uparrow} = u_{\mathbf{k}} \mathbf{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \mathbf{c}^\dagger_{-\mathbf{k}\downarrow} \tag{1}$$

$$\mathbf{d}_{\mathbf{k}\downarrow} = u_{\mathbf{k}} \mathbf{c}_{\mathbf{k}\downarrow} - v_{\mathbf{k}} \mathbf{c}^\dagger_{-\mathbf{k}\uparrow}, \tag{2}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are real and even functions of $\mathbf{k}$, i.e., $v_{-\mathbf{k}} = v_{\mathbf{k}}$ and $u_{-\mathbf{k}} = u_{\mathbf{k}}$. What are the requirements on $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ so that the new operators $\mathbf{d}^\dagger_{\mathbf{k}\sigma}$ and $\mathbf{d}_{\mathbf{k}\sigma}$ can be identified with fermionic creation and annihilation operators?

b) Consider the Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} \mathbf{c}^\dagger_{\mathbf{k}\sigma} \mathbf{c}_{\mathbf{k}\sigma} - \Delta \sum_{\mathbf{k}} \left(\mathbf{c}^\dagger_{\mathbf{k}\uparrow} \mathbf{c}^\dagger_{-\mathbf{k}\downarrow} + \mathbf{c}_{-\mathbf{k}\downarrow} \mathbf{c}_{\mathbf{k}\uparrow} \right) \tag{3}$$

Use the transformation of part (a) to diagonalize the Hamiltonian, i.e., to write it in the form $\mathcal{H} = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \mathbf{d}^\dagger_{\mathbf{k}\sigma} \mathbf{d}_{\mathbf{k}\sigma} + \text{const.}$. What is the eigenenergy $E_{\mathbf{k}}$?