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Advanced Quantum Mechanics

Exercise sheet 7

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **November 24th**, 2014 (10 am, i.e. before the lecture starts)

17. Coherent states of the electromagnetic field (6 points)

In this exercise we consider coherent states of the electromagnetic field and derive some of their important properties. Coherent states are eigenstates of the annihilation operator $a_{\mathbf{k},\lambda}$ for fixed $\mathbf{k}$ and λ:

$$a_{\mathbf{k},\lambda}|\phi\rangle = \phi|\phi\rangle$$

for an arbitrary complex number ϕ.

a) Show that $|\phi\rangle$ is a coherent state:

$$|\phi\rangle = c \exp\left(\phi\, a^{\dagger}_{\mathbf{k},\lambda}\right)|\Omega\rangle,$$

where $|\Omega\rangle$ is the photon vacuum. Determine the normalization constant c by requiring the coherent state to be normalized. Compute the overlap of two different coherent states $\langle\theta|\phi\rangle$.

b) Show that the action of the creation operator on a coherent state $|\phi\rangle$ is given by

$$a^{\dagger}_{\mathbf{k},\lambda}|\phi\rangle = (\partial_\phi + \phi^\star/2)|\phi\rangle.$$

c) Show that coherent states form a complete set, i.e. show that

$$\frac{1}{\pi}\int d\phi \int d\bar{\phi}\ |\phi\rangle\langle\phi| = \mathbb{1},$$

where $\int d\phi \int d\bar{\phi}$ denotes the integration over the complex plane and $\mathbb{1}$ is the identity operator.
Hint: One possible (though not the most elegant) way of proofing this identity is to show that the left-hand-side acts as the identity operator on the eigenbasis of the number operator.

d) One important property of coherent states is that they are the closest possible analog to classical states. Compute the expectation values $\langle\phi|\mathbf{E}(\mathbf{r},t)|\phi\rangle$ and $\langle\phi|\mathbf{B}(\mathbf{r},t)|\phi\rangle$ and compare to the behavior of classical electromagnetic fields.

18. Casimir effect (14 points)

In the lectures, you found that the quantization of the electromagnetic field leads to an infinite zero-point energy $E_0 = \frac{1}{2}\sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k}}$, which is usually discarded as being unphysical. While the zero-point energy itself cannot be measured, *differences* in the zero-point energy can be measured and have physical consequences. One important example is the Casimir effect, which is the attractive force between two uncharged, parallel metallic plates in vacuum. In this exercise, we reproduce Casimir's original calculation of this effect.

a) We consider a cavity of dimension $L_x \times L_y \times L_z$ with metallic walls. From classical electrodynamics we know that the allowed modes in the cavity have wave vectors $k_i = n\frac{\pi}{L_i}$ for $i = x, y, z$ and $n \in \mathbb{N}_0$. For each $\mathbf{k}$ there are two possible polarizations, except when one (or two) of the momentum components is zero, in which case there is only one allowed polarization. Show that the zero-point energy for such a cavity can for $L_x \ll L_y, L_z$ (L_y, L_z are considered very large) be approximated by:

$$W_{L_x} = \sum_{\mathbf{k}} \frac{1}{2}\hbar\omega_{\mathbf{k}} = \hbar c \frac{L_y L_z}{\pi^2} \int_0^\infty dk_x \int_0^\infty dk_y \left(\frac{1}{2}\sqrt{k_y^2 + k_z^2} + \sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{L_x}\right)^2 + k_y^2 + k_z^2} \right).$$

b) Consider now a cubic cavity of dimensions $L \times L \times L$. We want to compute the difference in the zero-point energy for placing an *additional* conducting wall of dimensions $L \times L$ at position $x = R \ll L$ versus placing it at position $x = L/2$. Using the results of a) show that the difference δE can be written as:

$$\begin{aligned} \delta E = \hbar c \frac{L^2}{\pi^2} \int_0^\infty dk_y \int_0^\infty dk_z \left(\frac{1}{2}\sqrt{k_y^2 + k_z^2} + \sum_{n=1}^{\infty} \sqrt{\left(\frac{n\pi}{R}\right)^2 + k_y^2 + k_z^2} \right) \\ - \hbar c \frac{L^2 R}{\pi^3} \int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z \sqrt{k_x^2 + k_y^2 + k_z^2}. \end{aligned}$$

This expression is still divergent. In order to regularize it we multiply the integrand with a smooth function $f(|\mathbf{k}|/k_c)$ such that $f(0) = 1$ and $f(|\mathbf{k}|/k_c) \to 0$ for $|\mathbf{k}| \gg k_c$, where k_c is a (large) momentum cutoff. This is more than just a mathematical artifact: the conducting plate does not affect the electromagnetic fields for sufficiently high frequencies, thus large momenta do not contribute to the energy difference. The exact form of the function f is not important for this calculation. In particular, the end result will not depend on f itself, but only the limiting behavior at $|\mathbf{k}| \ll k_c$ and $|\mathbf{k}| \gg k_c$.

c) You can simplify the expression of δE by introducing polar coordinates, $k_y = \kappa\cos(\alpha)$ and $k_z = \kappa\sin(\alpha)$ and performing the integral over α. Proceed by substituting $u = n^2 + \left(\frac{R\kappa}{\pi}\right)^2$ in order to move the n dependence from the integrand to the domain boundary of the integral. A similar substitution should be done for the k_x integration. The energy difference can then be written in the form $\delta E \sim \frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dn\, F(n)$, which can be approximated using the Euler-Maclaurin Formula:

$$\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^{\infty} dn\, F(n) = -\frac{1}{6\cdot 2!}F'(0) + \frac{1}{30\cdot 4!}F^{(3)}(0) - \frac{1}{42\cdot 6!}F^{(5)}(0) + \ldots,$$

where we assumed that the function F as well as all its derivatives $F^{(n)}$ vanish at infinity. Neglect the terms $F^{(n)}$ with $n \geq 5$ (argue under which conditions this approximation is valid) to obtain the final result $\delta E = -\hbar c \frac{\pi^2}{720}\frac{L^2}{R^3}$. Why does this imply an attractive force between two parallel conducting plates?