

Advanced Quantum Mechanics

Exercise sheet 12

Winter term 2014/15

Homepage: <http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml>

Due date: None. The solution is discussed in the tutorials on Wednesday, January 7th.

30. Quantization of the Dirac equation

In this exercise, we will work through the second quantisation of the Dirac fields.

a) Show that

$$\psi(\mathbf{r}, t) = \sum_{\mathbf{k}, \sigma} \sqrt{\frac{m_0 c^2}{E_{\mathbf{k}} V}} \left(b_{\sigma, \mathbf{k}} u_{\sigma}(\mathbf{k}) e^{-i\omega t + i\mathbf{k}\mathbf{r}} + d_{\sigma, \mathbf{k}}^* w_{\sigma}(\mathbf{k}) e^{i\omega t - i\mathbf{k}\mathbf{r}} \right)$$

with $E_{\mathbf{k}} = \sqrt{(\hbar \mathbf{k} c)^2 + (m_0 c^2)^2}$ and

$$u_r(\mathbf{k}) = \sqrt{\frac{E_{\mathbf{k}} + m_0 c^2}{2m_0 c^2}} \begin{pmatrix} \chi_r \\ \frac{c\hbar}{E_{\mathbf{k}} + m_0 c^2} \boldsymbol{\sigma} \cdot \mathbf{k} \chi_r \end{pmatrix}$$

$$w_r(\mathbf{k}) = -\sqrt{\frac{E_{\mathbf{k}} + m_0 c^2}{2m_0 c^2}} \begin{pmatrix} \frac{c\hbar}{E_{\mathbf{k}} + m_0 c^2} \boldsymbol{\sigma} \cdot \mathbf{k} (i\sigma^2 \chi_r) \\ (i\sigma^2 \chi_r) \end{pmatrix}$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is indeed a solution of the Dirac equation with the γ -matrices in the Dirac-Pauli representation.

b) Further show that

$$\bar{u}_{\sigma}(\mathbf{k}) \gamma^0 u_{\sigma'}(\mathbf{k}) = \bar{w}_{\sigma}(\mathbf{k}) \gamma^0 w_{\sigma'}(\mathbf{k}) = \frac{E_{\mathbf{k}}}{m_0 c^2} \delta_{\sigma, \sigma'}$$

$$\bar{u}_{\sigma}(\mathbf{k}) \gamma^0 w_{\sigma'}(-\mathbf{k}) = \bar{w}_{\sigma}(\mathbf{k}) \gamma^0 u_{\sigma'}(-\mathbf{k}) = 0.$$

- c) In analogy to the quantization of the Klein-Gordon field, we now replace the complex factors $b_{\sigma,\mathbf{k}}$ and $d_{\sigma,\mathbf{k}}^*$ by operators $b_{\sigma,\mathbf{k}} \rightarrow \hat{b}_{\sigma,\mathbf{k}}$ and $d_{\sigma,\mathbf{k}}^* \rightarrow \hat{d}_{\sigma,\mathbf{k}}^\dagger$. However, in contrast to the Klein-Gordon field we impose *anticommutation relations* for the operators. Use the result of part **b)** to derive the second quantized expression for the momentum, energy, and charge operator

$$\begin{aligned}
 P^j &= i\hbar \int d^3r \bar{\psi}(\mathbf{r}, t) \gamma^0 \partial^j \psi(\mathbf{r}, t) \\
 H &= i\hbar \int d^3r \bar{\psi}(\mathbf{r}, t) \gamma^0 \partial_t \psi(\mathbf{r}, t) \\
 Q &= q \int d^3r \bar{\psi}(\mathbf{r}, t) \gamma^0 \psi(\mathbf{r}, t).
 \end{aligned}$$

- d) Using the final expressions of part **c)**, explain why it was essential to impose anti-commutation relations for the operators. In particular, what goes wrong when you try to impose commutation relations? How is the vacuum defined and what difference do you see to the bosonic case?