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Advanced Quantum Mechanics

Exercise sheet 1

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

Due date: Monday, **October 26th**, 2015 (10 am, i.e. before the lecture starts)

3. Density operator for two spin-1/2 (10 points)

Consider two spin-$\frac{1}{2}$'s with a wavefunction $|\sigma\sigma'\rangle = |\sigma\rangle_1 \otimes |\sigma'\rangle_2$ where $\sigma, \sigma' = \uparrow, \downarrow$. These denote the z-component of the spin, i.e. $\mathbf{S}_1^z|\uparrow\rangle_1 = \frac{\hbar}{2}|\uparrow\rangle_1$, $\mathbf{S}_1^z|\downarrow\rangle_1 = -\frac{\hbar}{2}|\downarrow\rangle_1$ and similarly for the second spin.

a) The system is with a probability of 50% in a singlet state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ and with a probability of 50% in a triplet state $|\uparrow\uparrow\rangle$. What is the corresponding density operator?

b) Alice measures the first spin along the $z-$axis. What is the probability that after the measurement the second spin is in the state $|\uparrow\rangle_2$ and $|\downarrow\rangle_2$, respectively?

c) Show that the partial trace

$$\boldsymbol{\rho}_{\text{red}} = \text{Sp}_1\{\boldsymbol{\rho}\} = {}_1\langle\uparrow|\boldsymbol{\rho}|\uparrow\rangle_1 + {}_1\langle\downarrow|\boldsymbol{\rho}|\downarrow\rangle_1 \tag{1}$$

recovers the reduced density operator correctly. This means that it is independent of the measurement of Alice!

4. Commutators and Anticommutators (10 points)

a) Show that for operators $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ the following identities hold:

- $[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}$
- $[\mathbf{AB}, \mathbf{C}] = \mathbf{A}\{\mathbf{B}, \mathbf{C}\} - \{\mathbf{A}, \mathbf{C}\}\mathbf{B}$
- $\{\mathbf{AB}, \mathbf{C}\} = \mathbf{A}\{\mathbf{B}, \mathbf{C}\} - [\mathbf{A}, \mathbf{C}]\mathbf{B}$

b) Show that

$$e^{\mathbf{A}}\mathbf{B} = (\mathbf{B} + [\mathbf{A}, \mathbf{B}])e^{\mathbf{A}}$$

assuming that the commutator $[\mathbf{A}, \mathbf{B}]$ commutes with $\mathbf{A}$, i.e. $[\mathbf{A}, [\mathbf{A}, \mathbf{B}]] = 0$.
Hint: Expand the exponential and use induction for the proof.

c) Use your result of **b)** to show that

$$e^{\mathbf{A}}e^{\mathbf{B}} = e^{[\mathbf{A},\mathbf{B}]}e^{\mathbf{B}}e^{\mathbf{A}}$$

if the commutator $[\mathbf{A}, \mathbf{B}]$ also commutes with $\mathbf{B}$, i.e. $[\mathbf{B}, [\mathbf{A}, \mathbf{B}]] = 0$.
Comment: This last identity is a variant of the famous Baker-Campbell-Hausdorff formula, which you will encounter several more times in this course.