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Advanced Quantum Mechanics

Exercise sheet 9

Winter term 2015/16

22. Klein-Gordon equation (10 points)

The complex Klein-Gordon field is expressed in terms of creation operators as

$$\phi^\dagger(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_\mathbf{k} \sqrt{\frac{\hbar}{2\omega_\mathbf{k}}} \left(e^{i\omega_\mathbf{k} t - i\mathbf{k}\mathbf{r}} c_\mathbf{k}^\dagger + e^{-i\omega_\mathbf{k} t + i\mathbf{k}\mathbf{r}} b_\mathbf{k} \right) \tag{1}$$

where $c_\mathbf{k}^\dagger$ and $b_\mathbf{k}^\dagger$ are bosonic creation operators and $\hbar\omega_\mathbf{k} = \sqrt{(m_0 c^2)^2 + (\hbar\mathbf{k}c)^2}$. The Hamiltonian reads

$$\mathcal{H} = \sum_\mathbf{k} \hbar\omega_\mathbf{k} \left(c_\mathbf{k}^\dagger c_\mathbf{k} + b_\mathbf{k}^\dagger b_\mathbf{k} + 1 \right). \tag{2}$$

The conjugate momentum density to the Klein-Gordon field is given by

$$\pi(\mathbf{r}, t) = \partial_t \phi^\dagger(\mathbf{r}, t). \tag{3}$$

a) Use the Heisenberg equations of motion to express $\pi(\mathbf{r}, t)$ in terms of $c_\mathbf{k}$ and $b_\mathbf{k}$. Show that the equal-time commutators are given by $[\phi^\dagger(\mathbf{r}, t), \phi(\mathbf{r}', t)] = [\pi^\dagger(\mathbf{r}, t), \pi(\mathbf{r}', t)] = 0$ and $[\pi(\mathbf{r}, t), \phi(\mathbf{r}', t)] = -i\hbar\delta(\mathbf{r} - \mathbf{r}')$.

b) Express the creation operators $c_\mathbf{k}^\dagger$ and $b_\mathbf{k}^\dagger$ in terms of $\phi^\dagger(\mathbf{r}, t)$ and $\pi(\mathbf{r}, t)$ using the inverse Fourier transform.

c) Using the result of **b)** show that the Hamiltonian can be written as

$$\mathcal{H} = \int d\mathbf{r} \left(\pi^\dagger \pi + c^2 (\boldsymbol{\nabla}\phi^\dagger)(\boldsymbol{\nabla}\phi) + \frac{m_0^2 c^4}{\hbar^2} \phi^\dagger \phi \right). \tag{4}$$

23. Real Klein-Gordon field (4 points)

In the lectures, you considered the complex Klein-Gordon field and derived its second quantized expression. Here, we consider the real Klein-Gordon field, which upon quantization becomes a Hermitian operator $\phi^\dagger(\mathbf{r}, t) = \phi(\mathbf{r}, t)$.

a) Show that $\phi^\dagger(\mathbf{r}, t) = \phi(\mathbf{r}, t)$ implies that $\phi(\mathbf{r}, t)$ is of the form

$$\phi(\mathbf{r}, t) = \sqrt{\frac{\hbar}{V}} \sum_\mathbf{k} \frac{1}{\sqrt{2\omega_K}} \left(e^{i\omega_\mathbf{k} t - i\mathbf{k}\cdot\mathbf{r}} c_\mathbf{k}^\dagger + e^{-i\omega_\mathbf{k} t + i\mathbf{k}\cdot\mathbf{r}} c_\mathbf{k} \right)$$

What consequences has $\phi^\dagger(\mathbf{r}, t) = \phi(\mathbf{r}, t)$ for the charge of the field operator?

b) Derive the Hamiltonian H and the momentum P in terms of the creation and annihilation operators.

24. Parity operator of the real Klein-Gordon field (6 points)

The parity transformation (i.e. space inversion) is defined by

$$\phi(\mathbf{r}, t) \to \mathcal{P}\phi(\mathbf{r}, t)\mathcal{P}^{-1} \equiv \eta_P \phi(-\mathbf{r}, t) \qquad (*)$$

where the parity operator $\mathcal{P}$ is a unitary operator which leaves the vacuum invariant $\mathcal{P}|0\rangle = |0\rangle$, and $\eta_P = \pm 1$ is called the *intrinsic* parity of the field.

a) Show that the parity transformation leaves the Hamiltonian density $\mathcal{H}$

$$\mathcal{H}(\mathbf{r}, t) = \frac{1}{2}\left[(\partial_t \phi)^2 + c^2(\vec{\nabla}\phi)^2 + c^2\mu^2\phi^2\right]$$

invariant and, thus, describes a symmetry of our system.

b) Show that
$$\mathcal{P}|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = \eta_P^n| - \mathbf{k}_1, \ldots, -\mathbf{k}_n\rangle,$$
where $|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = \prod_{j=1}^{n} c_{\mathbf{k}_j}^\dagger |0\rangle$.

c) Prove that the parity operator is given by $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2$, where

$$\mathcal{P}_1 = \exp\left[-i\frac{\pi}{2}\sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}}\right], \qquad \mathcal{P}_2 = \exp\left[i\frac{\pi}{2}\eta_P \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}\right].$$

To this end, show that $\mathcal{P}_1\mathcal{P}_2$ is a unitary operator that leaves the vacuum invariant and satisfies $(*)$.
Hint: In order to prove $(*)$ it is useful to first derive

$$\mathcal{P}_1 c_{\mathbf{k}} \mathcal{P}_1^{-1} = i c_{\mathbf{k}}, \qquad\qquad \mathcal{P}_2 c_{\mathbf{k}} \mathcal{P}_2^{-1} = -i\eta_P c_{-\mathbf{k}}.$$