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Advanced Quantum Mechanics

Exercise sheet 10

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

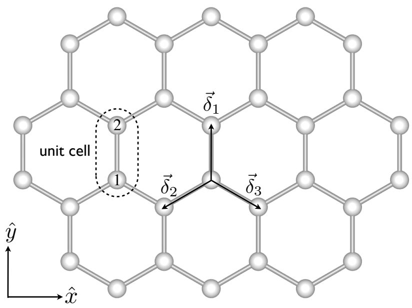
Due date: Monday, **January 11th**, 2016 (10 am, i.e. before the lecture starts)

25. Dirac fermions (10 points)

The Dirac equation is usually associated with high-energy physics. However, in recent years more and more examples of condensed matter systems have emerged that are described by (variants) of the Dirac equation for sufficiently *low* energies. The most prominent example is Graphene – a single sheet of graphite – whose experimental realization was awarded the Nobel prize in 2010. The band structure of Graphene is obtained by considering electrons hopping on a honeycomb lattice, where each site (at position $\vec{r}$) has three nearest neighbors located at $\vec{r} + \vec{\delta}_j$, $j = 1, 2, 3$ with

$$\vec{\delta}_1 = \frac{a}{2}(-\sqrt{3}, -1) \qquad \vec{\delta}_2 = \frac{a}{2}(\sqrt{3}, -1) \qquad \vec{\delta}_3 = a(0, 1).$$

where a denotes the lattice constant. The translation vectors are given by $\vec{t}_1 = \vec{\delta}_3 - \vec{\delta}_1$ and $\vec{t}_2 = \vec{\delta}_3 - \vec{\delta}_2$.



For simplicity, we consider spinless fermions and an isotropic hopping amplitude t:

$$\hat{H} = -t \sum_{\vec{r}} \sum_{j=1}^{3} c^\dagger(\vec{r}) c(\vec{r} + \delta_j). \tag{1}$$

a) With the use of Fourier transformations, compute the energy spectrum of the Hamiltonian (1) and determine the ground state.

Comment: Note that there are two sites per unit cell for the honeycomb lattice, denoted by 1 and 2. The fermionic creation and annihilation operators can thus be labeled by the unit cell position $\vec{R} = n_1 \vec{t}_1 + n_2 \vec{t}_2$ as well as the sublattice index 1 and 2. The Fourier transform of these operators is then given by

$$c_j(\vec{R}) = \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k}\cdot\vec{R}} c_j(\vec{k}) \qquad c_j^\dagger(\vec{R}) = \int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot\vec{R}} c_j^\dagger(\vec{k})$$

Why can you ignore the separation of the sites 1 and 2 within the unit cell in the Fourier transformation?

b) The energy spectrum of **a)** is gapless at two points $\vec{K} = \frac{1}{3}\vec{q}_1 + \frac{2}{3}\vec{q}_2$ and $\vec{K}' = \frac{2}{3}\vec{q}_1 + \frac{1}{3}\vec{q}_2$, with $\vec{q}_j$ denoting the reciprocal lattice vectors defined by $\vec{q}_i \cdot \vec{t}_j = 2\pi\delta_{i,j}$. Expand the Hamiltonian around these gapless points in small momentum deviations $\vec{q} = \vec{k} - \vec{K}$ (respectively $\vec{q} = \vec{k} - \vec{K}'$) and show that in the low-energy approximation, $\hat{H}$ reduces to the sum of two two-dimensional Dirac Hamiltonians, i.e. the terms are of the form

$$v_F(q_x\sigma_x + q_y\sigma_y)$$

with the velocity of light replaced by the Fermi velocity.

c) The two-dimensional Dirac equation in the previous part describes a massless particle with a linear dispersion. What additional term in the Hamiltonian would make this particle massive, i.e. induce a gap in the energy dispersion?

26. Weyl fermions (6 points)

The second species of fermions we want to look at are Weyl fermions – chiral fermions that arise quite naturally in the chiral representation of the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \text{and } \gamma^n = \begin{pmatrix} 0 & -\sigma^n \\ \sigma^n & 0 \end{pmatrix}$$

where σ^n are Pauli matrices for $n = 1, 2, 3$.

a) Assume that $\Psi(x)$ is a solution of the Dirac equation with the γ-matrices in the Weyl representation. Under which conditions is $\hat{\Psi}(x) = \exp[i\alpha\gamma^5]\Psi(x)$ also a solution of the Dirac equation?

b) Write the Dirac equation for the two-component spinors $\Psi_1(x)$ and $\Psi_2(x)$ with $\Psi(x) = (\Psi_1(x), \Psi_2(x))$. Show that the massless Dirac equation decomposes into two independent equations for $\Psi_1(x)$ and $\Psi_2(x)$.

c) Using Fourier transformation show that

$$\Psi_R(x) = \frac{1}{2}\left(\mathbb{1} + \gamma^5\right)\Psi(x) = (\Psi_1(x), 0)^T$$
$$\Psi_L(x) = \frac{1}{2}\left(\mathbb{1} - \gamma^5\right)\Psi(x) = (0, \Psi_2(x))^T$$

are eigenstates to the helicity operator

$$h_{\mathbf{k}} = \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{|\mathbf{k}|},$$

i.e. they carry a momentum $\mathbf{k}$ that is either parallel or antiparallel to their spin orientation.

Weyl fermions have recently attracted a wave of interest in condensed matter systems. A pedagogical review of this phenomenon can be found at http://physics.aps.org/articles/v4/36.

27. Majorana fermions (4 points)

Majorana fermions are probably the most exotic species of fermions – they describe (chargeless) fermions that are their own antiparticle. In high-energy physics they are used to discuss the nature of the neutrino, which is speculated to be its own antiparticle. You have already seen a condensed matter realization as well in exercise **14** – the Majorana chain. Majorana fermions are the simplest particles that allow for 'topologically protected quantum computing' – a quantum computing scheme that relies on the entangled nature of such Majorana fermions to keep the Qbits coherent. A pedagogical review of the quest for quantum computers can be found at http://www.nature.com/news/physics-quantum-computer-quest-1.16457.
Let us consider the Majorana representation of the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \text{ and } \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}.$$

a) Assume that $\Psi(x)$ is a solution of the Dirac equation with the γ-matrices in the Majorana representation. Show that the complex conjugate $\Psi^\star(x)$ is then also a solution of the Dirac equation.

b) Show that in the Majorana representation, we can write the complex spinor in terms of two real components $\Psi(x) = \Psi_1 + i\Psi_2$ with $\Psi_j^\star = \Psi_j$, such that the Dirac equation decomposes into two separate equations for Ψ_1 and Ψ_2, respectively.

c) Part **b)** implies that if the γ matrices are in the Majorana representation it is sufficient to consider real solutions of the Dirac equation, so-called Majorana fermions. Show that the condition $\Psi^\star = \Psi$ implies that $\bar{\Psi}\Psi = 0$.
Remark: This property is no longer valid for anti-commuting $\bar{\Psi}$ and Ψ.