Institut für Theoretische Physik
Universität zu Köln

Prof. Dr. S. Trebst
Dr. M. Hermanns

Advanced Quantum Mechanics

Exercise sheet 11

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml
Due date: Monday, **January 18th**, 2016 (10 am, i.e. before the lecture starts)

28. Quantization of the Dirac equation (20 points)

In this exercise, we will work through the second quantisation of the Dirac fields.

a) Show that

$$\psi(\mathbf{r},t) = \sum_{\mathbf{k},\sigma} \sqrt{\frac{m_0c^2}{E_\mathbf{k}V}} \left(b_{\sigma,\mathbf{k}} u_\sigma(\mathbf{k}) e^{-i\omega t + i\mathbf{k}\mathbf{r}} + d^\star_{\sigma,\mathbf{k}} w_\sigma(\mathbf{k}) e^{i\omega t - i\mathbf{k}\mathbf{r}} \right)$$

with $E_\mathbf{k} = \sqrt{(\hbar\mathbf{k}c)^2 + (m_0c^2)^2}$ and

$$u_r(\mathbf{k}) = \sqrt{\frac{E_\mathbf{k} + m_0c^2}{2m_0c^2}} \begin{pmatrix} \chi_r \\ \frac{c\hbar}{E_\mathbf{k}+m_0c^2} \boldsymbol{\sigma}\cdot\mathbf{k}\,\chi_r \end{pmatrix}$$

$$w_r(\mathbf{k}) = -\sqrt{\frac{E_\mathbf{k} + m_0c^2}{2m_0c^2}} \begin{pmatrix} \frac{c\hbar}{E_\mathbf{k}+m_0c^2} \boldsymbol{\sigma}\cdot\mathbf{k}\,\left(i\sigma^2\chi_r\right) \\ \left(i\sigma^2\chi_r\right) \end{pmatrix}$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is indeed a solution of the Dirac equation with the γ-matrices in the Dirac-Pauli representation.

b) Further show that

$$\bar{u}_\sigma(\mathbf{k})\gamma^0 u_{\sigma'}(\mathbf{k}) = \bar{w}_\sigma(\mathbf{k})\gamma^0 w_{\sigma'}(\mathbf{k}) = \frac{E_\mathbf{k}}{m_0c^2}\delta_{\sigma,\sigma'}$$

$$\bar{u}_\sigma(\mathbf{k})\gamma^0 w_{\sigma'}(-\mathbf{k}) = \bar{w}_\sigma(\mathbf{k})\gamma^0 u_{\sigma'}(-\mathbf{k}) = 0\,.$$

c) In analogy to the quantization of the Klein-Gordon field, we now replace the complex factors $b_{\sigma,\mathbf{k}}$ and $d^{\star}_{\sigma,\mathbf{k}}$ by operators $b_{\sigma,\mathbf{k}} \to \hat{b}_{\sigma,\mathbf{k}}$ and $d^{\star}_{\sigma,\mathbf{k}} \to \hat{d}^{\dagger}_{\sigma,\mathbf{k}}$. However, in contrast to the Klein-Gordon field we impose *anticommutation relations* for the operators. Use the result of part **b)** to derive the second quantized expression for the momentum, energy, and charge operator

$$
\begin{aligned}
P^j &= i\hbar \int d^3r\, \bar{\psi}(\mathbf{r},t)\gamma^0\partial^j\psi(\mathbf{r},t) \\
H &= i\hbar \int d^3r\, \bar{\psi}(\mathbf{r},t)\gamma^0\partial_t\psi(\mathbf{r},t) \\
Q &= q \int d^3r\, \bar{\psi}(\mathbf{r},t)\gamma^0\psi(\mathbf{r},t).
\end{aligned}
$$

d) Using the final expressions of part **c)**, explain why it was essential to impose anti-commutation relations for the operators. In particular, what goes wrong when you try to impose commutation relations? How is the vaccuum defined and what difference do you see to the bosonic case?