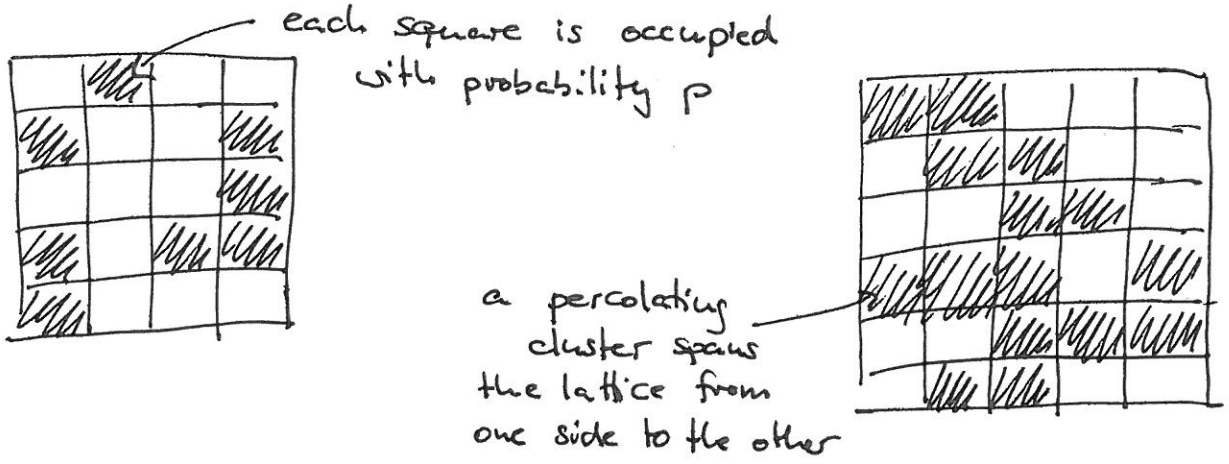
Percolation I

Ref: D. Stauffer and A. Aharony
Introduction to percolation theory
Taylor & Francis (1994)

The first problem that we will consider in this course is a conceptually very simple and purely geometric topic — percolation.

To introduce this topic, which we can consider to be one of the simplest classical many-body problems that does not even involve any dynamics, we consider site percolation on the square lattice.



In the infinite lattice limit, there is a sharp transition at a critical probability p_c: For $p < p_c$ there never is a percolating cluster, while for $p > p_c$ there always is a percolating cluster.

Percolation theory deals with questions like

- What is the critical probability p_c?
- How do the number of clusters and average cluster size depend on p?
- What is the probability that a site is on the percolating cluster?

- How does a finite lattice size change the results?
- How do the results depend on the lattice structure and on the specific percolation model used?

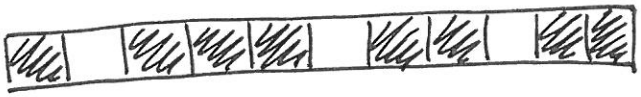
As we will see the answer to the last question will be that close to the critical probability p_c the properties depend only on the dimensionality d and on the type (lattice or continuum) but not on the specific lattice structure or percolation model. Thus, our results obtained for one lattice are "universal" in the sense that they will also apply to all other percolation models (of the same dimension / type).

Some examples:

- oil fields
- forest fires
- spread of diseases
- vulneration of the internet
- gelation of liquids
- baking of cookies

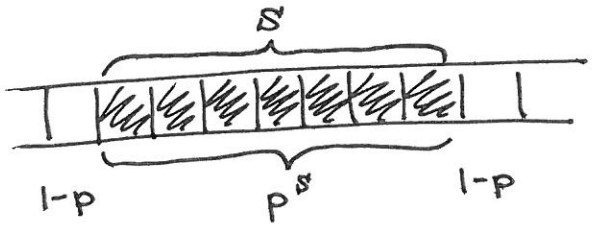
Exact solution in one dimension

In one spatial dimensions percolation can be solved exactly.



A percolating cluster spans the whole chain, not allowing a single empty site. Thus, $\boxed{p_c = 1}$.
As a consequence we can only study the system below the percolation transition, i.e. $p < p_c$, in one dimension.

Let's consider a cluster of size s



The probability that a given site is the left edge of a cluster of this finite size simply is

$$n_s = (1-p)^2 \cdot p^s \quad .$$

The probability that a random site is anywhere on an s-site cluster is $s \cdot n_s$.

The probability that an arbitrary site belong to any cluster is equal to the probability p that it is occupied

$$\sum_s n_s \cdot s = p \quad ,$$

where the sum runs from $s=1$ to $s=\infty$.

Let's show this explicitly

$$\sum_s n_s \cdot s = \sum_s (1-p)^2 \cdot p^s \cdot s = (1-p)^2 \sum_s p^s \cdot s$$

$$= (1-p)^2 \sum_s p \cdot \frac{d(p^s)}{dp} = (1-p)^2 p \frac{d(\sum_s p^s)}{dp}$$

$$= (1-p)^2 \cdot p \cdot \frac{d\left(\frac{p}{1-p}\right)}{dp} = p \quad \square$$

- In a similar form we can also estimate the average cluster size.

$$w_s = \frac{n_s \cdot s}{\sum_s n_s \cdot s}$$

w_s: prob. for an s-site cluster

$n_s \cdot s$: probability that an arbitrary site belongs to an s-cluster

$\sum_s n_s \cdot s$: prob. that it belongs to any finite cluster

The <u>average cluster size</u> S thus is

$$S = \sum_s w_s \cdot s = \sum_s \frac{n_s \cdot s^2}{\left(\sum_s n_s \cdot s\right)} = p$$

For the nominator we use the same trick as above, and find

$$(1-p)^2 \sum_s s^2 \cdot p^s = (1-p)^2 \left(p \frac{d}{dp}\right)^2 \sum_s p^s$$

$$= p \cdot \frac{1+p}{1-p}$$

We thus arrive at an average cluster size

$$S = \frac{1+p}{1-p}$$ → diverges for $p \to p_c = 1$

for $p < p_c$ in one dimension

We can further define the correlation function $g(r)$ (or pair connectivity) as the probability that a site at distance r apart from an occupied site belongs to the same cluster.

$$g(r=0) = 1$$

For a site at distance r all intermediate sites need to be occupied, which happens with probability

$$g(r) = p^r$$

For $p<1$ this correlation function goes to zero exponentially

$$g(r) = \exp(-r/\xi) \,,$$

where ξ is the correlation length

$$\xi = -\frac{1}{\ln p} \simeq \frac{1}{p_c - p}$$

→ also diverges for $p \to p_c = 1$

$\ln(1-x) = -x$

$\ln(p) = \ln(1-(1-p)) = -(1-p)$.

In particular, we see that average cluster size and correlation length are varied in the same way as $p \to p_c$

$$\boxed{S \propto \xi \quad \text{for } p \to p_c} \,.$$

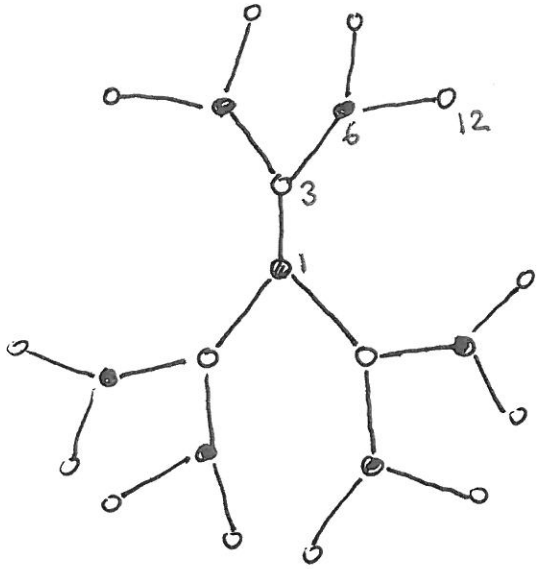
This is not generally true in higher dimensions, but the following sum rule will hold

$$\boxed{\sum_r g(r) = S \,.}$$

Exact solution in infinite dimensions

Besides the one-dimensional case, percolation can also be solved in a particular case, on the so-called Bethe lattice (or Cayley tree), which corresponds to infinite dimensionality.

- The Bethe lattice



$$g(r=0) = 1$$
$$g(r=1) = z$$
$$g(r=2) = z(z-1)$$
$$g(r=3) = z(z-1)^2$$

$$g(r) = g(r-1)(z-1)$$

The volume of the Bethe lattice is thus given by

$$V = \sum_{r=0}^{R} g(r) = 1 + z \cdot \frac{(z-1)^R - 1}{z-1-1} \approx z \cdot \frac{(z-1)^R}{z-2}$$

The boundary of the Bethe lattice is given by

$$\partial V = g(R) = z \cdot (z-1)^{R-1}$$

so that we have

$$\partial V \sim V \cdot \underbrace{\frac{z-2}{z-1}}_{\approx \text{const.}}$$

Since in d dimensions we have $V \sim L^d$ and $\partial V \sim L^{d-1}$, thus

$$\partial V \sim V^{1-1/d}$$

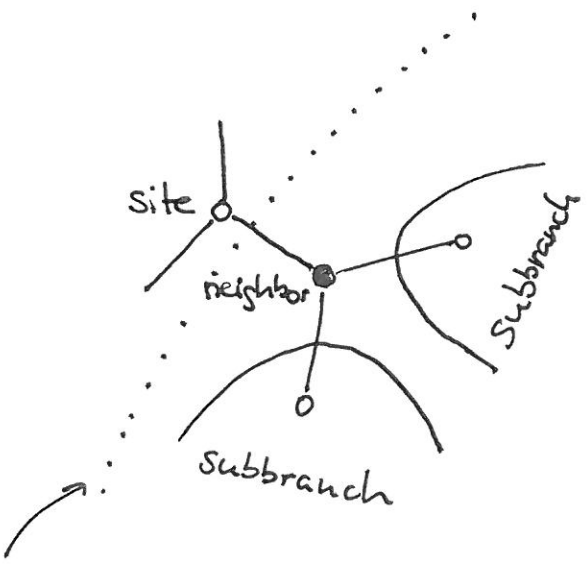
we conclude $\boxed{d=\infty}$ for the Bethe lattice

- Since the Bethe lattice has no loops, we can estimate the <u>percolation threshold</u> directly. To this end we follow an outward branch in the cluster – each site connects to z sites, including $z-1$ new branches, of which $p\cdot(z-1)$ are occupied. If $p\cdot(z-1)<1$ the number of branches decreases and the cluster will be finite. If $p\cdot(z-1)>1$ the number of branches increases and the cluster will be infinite. So:

$$p_c = \frac{1}{z-1}$$

Let's (again) calculate the <u>probability P</u> that a site is <u>on the percolating cluster</u>. Let's do this for $z=3$, for which $p_c = 1/2$.

To do this, let's define the probability Q that an arbitrary site is <u>not</u> connected to infinity through a fixed branch originating from this site:



$$Q = (1-p) + p\cdot Q^2$$

This equation has two solutions: $Q=1$ or $Q=\frac{1-p}{p}$.

The probability $(p-P)$ that the origin is occupied, but not connected to infinity through one of its 3 branches is pQ^3. Thus

$$P = p(1-Q^3)$$

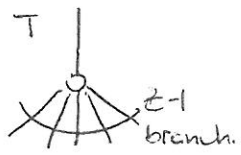
↖ prob. that a site is on percolating cluster

which gives $P=0$ for the solution $Q=1$, apparently belonging to $p < p_c = \frac{1}{2}$ and

$$P = p\left(1-\left(\frac{1-p}{p}\right)^3\right)$$

for the other solution, which corresponds to $p > p_c = \frac{1}{2}$.

Let's again turn to the <u>mean cluster size</u> S, for $p < p_c = \frac{1}{2}$. We can again make a recursive argument. Let us call (T) the <u>mean cluster size for one branch.</u> If the root of the branch is empty $T=0$, if it is occupied then the size is $1 + (z-1)\cdot T$, so we have



$$T = (1-p)\cdot 0 + p\cdot(1+2T)$$

with the solution $T = \frac{p}{1-2p}$ for $p < p_c = \frac{1}{2}$.

The size of the <u>total cluster</u> is the root plus the three branches

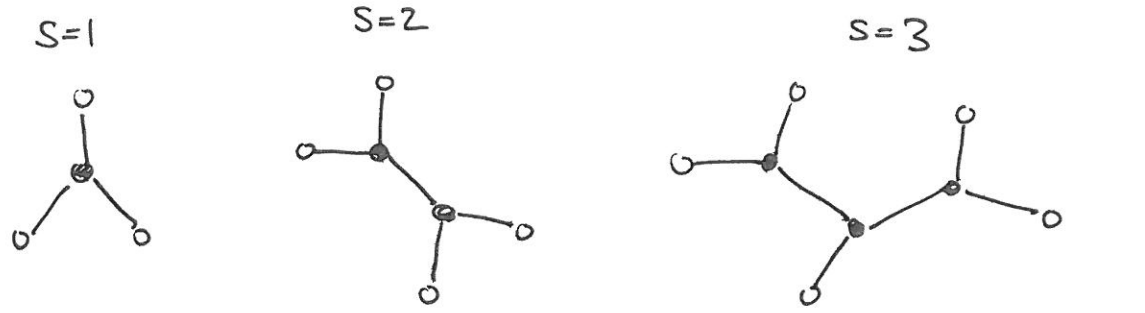
$$S = 1 + 3T = \frac{1+p}{1-2p}$$

Again we see that the mean cluster size diverges as we approach the percolation threshold

$$S \propto \frac{1}{p_c - p} .$$

Let's now turn to <u>cluster probabilities n_s, the average probability of clusters containing s sites.</u>

A cluster of s sites has $2+(z-2)\cdot s$ neighboring empty sites.



Let's now again set $z=3$ and look at the ratio

$$\frac{n_s(p)}{n_s(p_c)} = \left(\frac{p}{p_c}\right)^s \cdot \left(\frac{1-p}{1-p_c}\right)^{2+s}$$

$$= \left(\frac{1-p}{1-p_c}\right)^2 \cdot \left(\frac{p(1-p)}{p_c(1-p_c)}\right)^s .$$

$z=3$, $p_c=\frac{1}{2}$ ← exercise

$$= \left(\frac{1-p}{1-p_c}\right)^2 \cdot \left(1-4(p-p_c)^2\right)^s$$

$$\propto \exp(-c\cdot s)$$

with $c = -\ln\left(1-4(p-p_c)^2\right) \simeq (p-p_c)^2$ ($\ln(1-x) = -x$)

So, we have $n_s(p) = n_s(p_c)\cdot \exp(-c\cdot s)$.

All that is missing now is an expression for $n_s(p_c)$. Unfortunately, we cannot exactly calculate this, but make an "educated guess".

We have seen

$$S \propto \sum_s s^2 n_s$$

(S: mean cluster size)

For $p = p_c$ this sum is infinite, while for any other p it remains finite. So, $n_s(p_c)$ must decay faster than s^{-2}, but slower than s^{-3}. We make the ansatz

$$n_s(p_c) \simeq s^{-\tau}$$

for large s.

So let's calculate

$$\begin{aligned} S &\propto \sum_s s^2 n_s \\ &\propto \sum_s s^{2-\tau} \exp(-c \cdot s) \\ &\propto \int s^{2-\tau} \exp(-c \cdot s)\, ds \\ &= c^{\tau-3} \underbrace{\int z^{2-\tau} \exp(-z)\, dz}_{\text{some constant } \Gamma(3-\tau)} \qquad \text{s } z = c \cdot s \\ &\propto c^{\tau-3} \\ &\propto (p_c - p)^{2\tau-6} \end{aligned}$$

Since we earlier found $S \propto (p_c - p)^{-1}$, we obtain the so-called Fisher exponent via $2\tau - 6 = -1$:

$$\boxed{\tau = 5/2}$$

So, we have $n_s(p) = s^{-5/2} \exp(-c \cdot s)$ with $c \propto (p - p_c)^2$

Monte Carlo simulation

Monte Carlo simulations are probably the simplest numerical approach to investigate percolation on arbitrary lattices.

We will focus on three types of questions that can be addressed by MC simulations:

- What is the p-dependence of an arbitrary quantity X?
- What is the critical probability p_c?
- What are the values of the universal critical exponents?

X can be a quantity like the correlation length ξ, the average cluster size S, the probability that an occupied site is on the percolating cluster P or any other interesting observable.

For now we treat only finite systems of linear dimension $L < \infty$, the problem of extrapolating to $L \to \infty$ will be discussed later.

The expectation value of the quantity X can be calculated exactly on small lattices by a sum over all configurations of the system with $N = L^d$ sites:

$$\langle X \rangle = \sum_{n=0}^{N} \sum_{C \in C_{n,N}} p^n (1-p)^{N-n} X(C) ,$$

where $C_{n,N}$ is the set of configurations with n occupied sites in a lattice with N sites, and $X(C)$ is the value of the quantity X measured in the configuration C.

The number of terms in this sum increases exponentially like 2^N and can thus performed exactly only up to $N \approx 30$ sites.

Monte Carlo estimates

Larger lattices with up to $N = 10^6$ can no longer be done exactly, but Monte Carlo summation can provide estimates of this average to any desired accuracy.

To this end, the average over the complete set ~~is~~ of config's is replaced by a random sample of $M \approx 10^6 \ldots 10^{12}$ configurations c_i drawn randomly with the correct probability $p^n (1-p)^{N-n}$ for a configuration with n occupied sites.

To create configurations with the correct probabilities we draw a pseudo-random number u, uniformly distributed in $[0,1[$ for each site j. We set the site occupied if $u < p$ and empty otherwise.

The expectation value can be estimated by the sample mean:

$$\langle X \rangle \approx \overline{X} = \frac{1}{M} \sum_{i=1}^{M} X(c_i)$$

Since the sample size M is finite, the result is accurate only within statistical errors, estimated from the variance by

$$\Delta \overline{X} \approx \frac{1}{\sqrt{M}} \sqrt{\operatorname{Var} X} = \frac{1}{\sqrt{M}} \sqrt{\overline{X^2} - \overline{X}^2}$$

The convergence is only $O(\sqrt{M})$ and thus relatively slow. Still for large N it is much faster than the full summation.

Cluster labeling

We identify clusters using, for example, the Hoshen-Kopelman cluster labeling algorithm. It works as follows:

- allocate an array of size N to store the cluster label for each site

- loop through all sites i in the lattice.
  For occupied sites check if a cluster label has already been assigned to any of the neighboring sites.

- If no neighboring site is occupied or assigned a label, assign a new cluster label to this site.

- If one neighboring site is occupied and has a cluster label assigned, assign the same label to the site.

- If more than one neighbor site is occupied and the sites have different labels assigned, we have a problem. The current site connects two cluster parts which until now were disconnected and labeled as different clusters. We take the smallest of labels as the proper label and assign this to the current site.
  (For the other larger labels we would need to relabel all previously wrongly labeled sites.)

Trick: For each label we keep a list of the "proper" labels, initialized to the label itself.

To obtain a cluster label of a site we first obtain the assigned label. If its proper label agrees, we are done. Otherwise we replace the label by the proper label until the label agrees with its proper label

Scaling

We have seen that in both exactly solvable cases the quantities of interest have power-law singularities at p_c

The generalization of this to arbitrary lattices is the "scaling ansatz", which cannot be proven rigorously.

However, it can be motivated from

- the fractal behavior of the percolating cluster
- from renormalization group arguments
- from the good agreement of this ansatz with numerical results.

- The scaling ansatz

We generalize the scaling for the average cluster size S to

$$S \propto |p - p_c|^{-\gamma}$$

$\gamma = 1$ for $d = 1$
$\gamma = 1$ for $d = \infty$

where γ does not need to be equal to 1 in general.

For the correlation length ξ we define the exponent ν

$$\xi \propto |p - p_c|^{-\nu}$$

In the general case, we define the correlation ~~function~~ length as

$$\xi^2 = \frac{\sum_r r^2 g(r)}{\sum_r g(r)},$$

(Correlation function)

which is equivalent to the previous definition via exp. decay of G_r

the correlation function $g(r)$.

The pair correlation function $g(r)$ decays exponentially with the correlation length ξ for $p \neq p_c$.

At the critical concentration $p = p_c$, however, the correlation length ξ diverges and we assume a power law.

$$g(r) \propto r^{-(d-2+\eta)} \qquad (p = p_c) \qquad (*)$$

For the probability P (the strength of the percolating cluster), e.g. that a site is on the percolating cluster we make the ansatz

$$P \propto (p - p_c)^{\beta} \qquad (p > p_c)$$

Finally, we define an exponent for the cluster density M_0

$$M_0 = \sum_s n_s \qquad M_0 \propto |p - p_c|^{2-\alpha} .$$

Scaling relations

Fortunately, we do not have to calculate all five exponents, as there exist scaling relations between them.

To derive these, we postulate a scaling ansatz valid for all p and large s:

$$n_s(p) \propto s^{-\tau} \exp\left(-\frac{s}{s_\xi}\right) \qquad s \gg 1$$

(τ: Fisher exponent; s_ξ: cutoff cluster size (previous notation $c = 1/s_\xi$))

$$s_\xi \propto |p - p_c|^{-1/\sigma} \qquad p \to p_c$$

(σ: a 2nd "universal" exponent)

Note the role of $\boxed{s_\xi}$ as a <u>cluster cutoff size</u>, where $n_s(p)$ is characterized by

$$n_s(p) \propto \begin{cases} s^{-\tau} & \text{for } s \ll s_\xi \\ \text{decays rapidly} & \text{for } s \gg s_\xi \end{cases}$$

representing the crossover from a behavior of "critical clusters" (power-law distributed) to that of non-critical clusters.

Note that $n_s(p_c) = s^{-\tau}$ as $s_\xi = \infty$ for $p = p_c$.

For the Bethe lattice, we have previously calculated the average cluster size $S(p)$ already in general form

$$S(p) \propto \sum_{s=1}^{\infty} s^2 n_s(p)$$

$$= s_\xi^{3-\tau} \cdot \text{const} \qquad \text{const.} \quad \Gamma(3-\tau)$$

$$s_\xi = 1/c$$

with $\quad s_\xi \propto |p - p_c|^{-1/\beta} \quad$ for $p \to p_c$

we obtain $\quad S(p) \propto |p - p_c|^{\frac{\tau-3}{\beta}} \quad$ for $p \to p_c$

However, we just made the scaling ansatz

$$S(p) \propto |p - p_c|^{-\gamma} \quad \text{for } p \to p_c$$

which readily implies the scaling relation

$$\boxed{\gamma = \frac{3-\tau}{\beta}}$$

[Check for Bethe lattice $\gamma = 1 \quad \tau = 5/2 \quad \beta = 1/2 \quad \to$ OK.]

In a similar form, we obtain

$$\boxed{\beta = \frac{\tau-2}{\beta}}$$

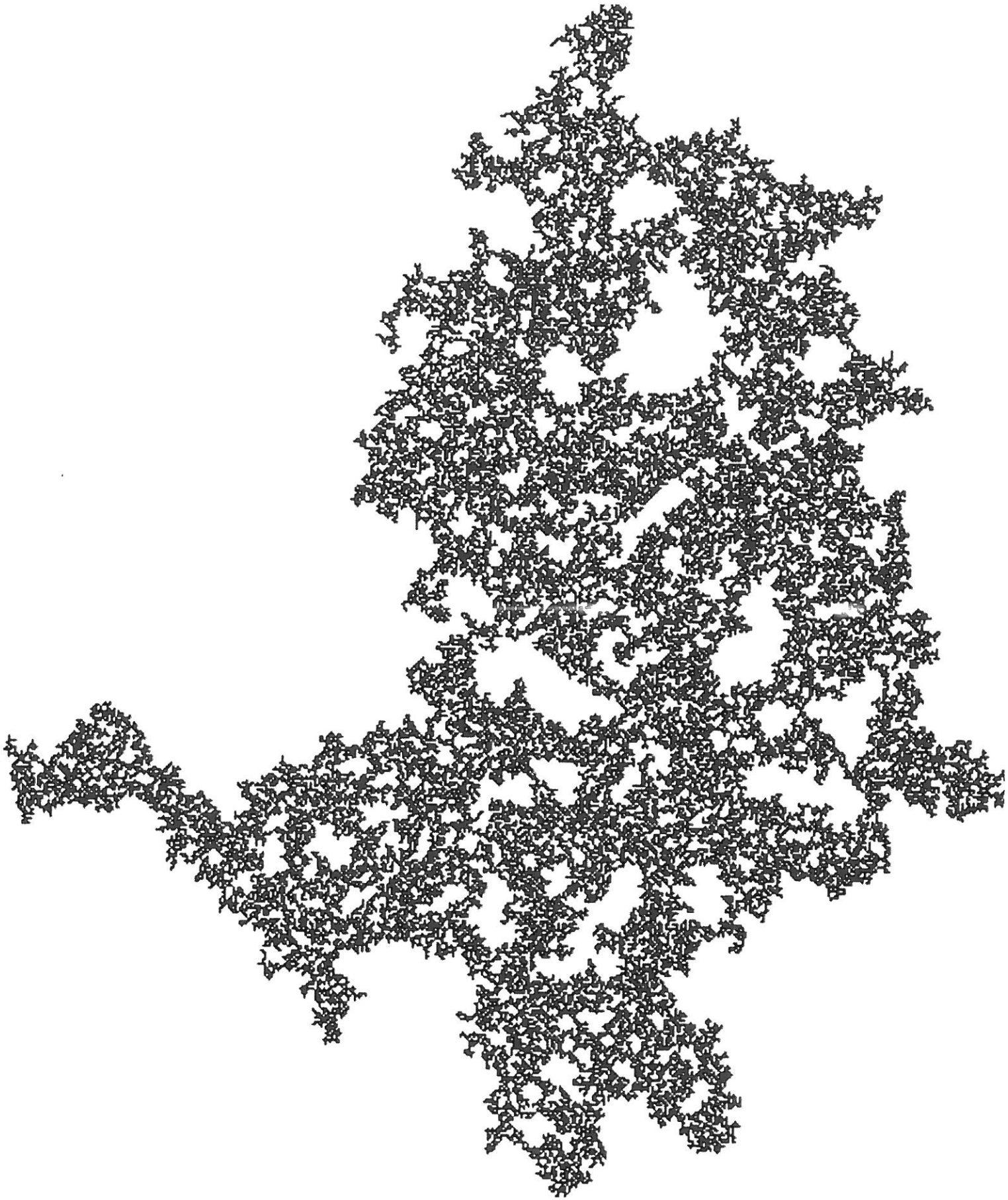
and

$$\boxed{2-\alpha = \frac{\tau-1}{\beta}}$$

.

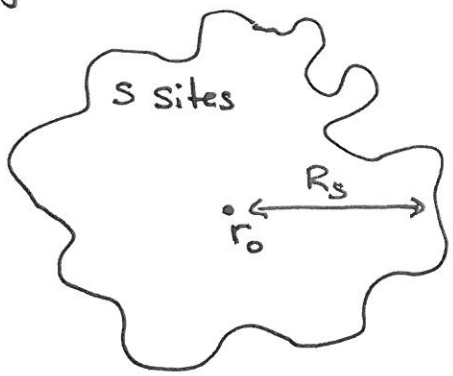
From which we can derive the scaling law

$$\boxed{2-\alpha = 2\beta + \gamma}$$



The percolating cluster at criticality is a complex object which appears self-similar on all length scales. Such an object is called a fractal.

Let's characterize this cluster object



- radius of gyration

$$R_S^2 = \sum_{i=1}^{S} \frac{|\vec{r}_i - \vec{r}_0|^2}{S}$$

where $\vec{r}_0 = \sum_{i=1}^{S} \frac{\vec{r}_i}{S}$ is the center of mass.

- this radius of gyration is related to the average distance between two cluster sites via

$$2R_S^2 = \sum_{ij} \frac{|\vec{r}_i - \vec{r}_j|^2}{S^2}$$

which can be derived for $r_0 = 0$ ($\rightarrow$ exercise).

- Self-similar objects are called fractals, since their dimension D defined by the relationship of volume to linear distance

$$V(R) \propto R^D$$

is a non-integer fraction D.

- For the percolating cluster we make the ansatz

$$\boxed{S \propto R_S^D \Leftrightarrow R_S \propto S^{1/D}}$$

We now want to connect this ansatz for the fractal dimensionality of the percolating cluster back to the correlation length ξ, thus establishing an intimate connection between self-similarity and the scaling ansatz.

$$\xi^2 = \frac{\sum_r r^2 g(r)}{\sum_r g(r)}$$

For a given cluster $2R_s^2$ is the average squared distance between two cluster sites, since a site belongs with probability $n_s \cdot s$ to an s-cluster, and since it is then connected to s sites, we can also calculate the correlation length as

$$\xi^2 = \frac{2 \sum_s R_s^2 s^2 n_s}{\sum_s s^2 n_s}$$

With $R_s \propto s^{1/D}$ and $\xi \propto |p - p_c|^{-\nu}$ one can derive

$$\boxed{D = \frac{\beta + \gamma}{\nu} = d - \frac{\beta}{\nu}}$$

show later

By integrating (*) one can also obtain the fractal dimension leading to another scaling law

$$\boxed{d - 2 + \eta = 2\beta/\nu}$$

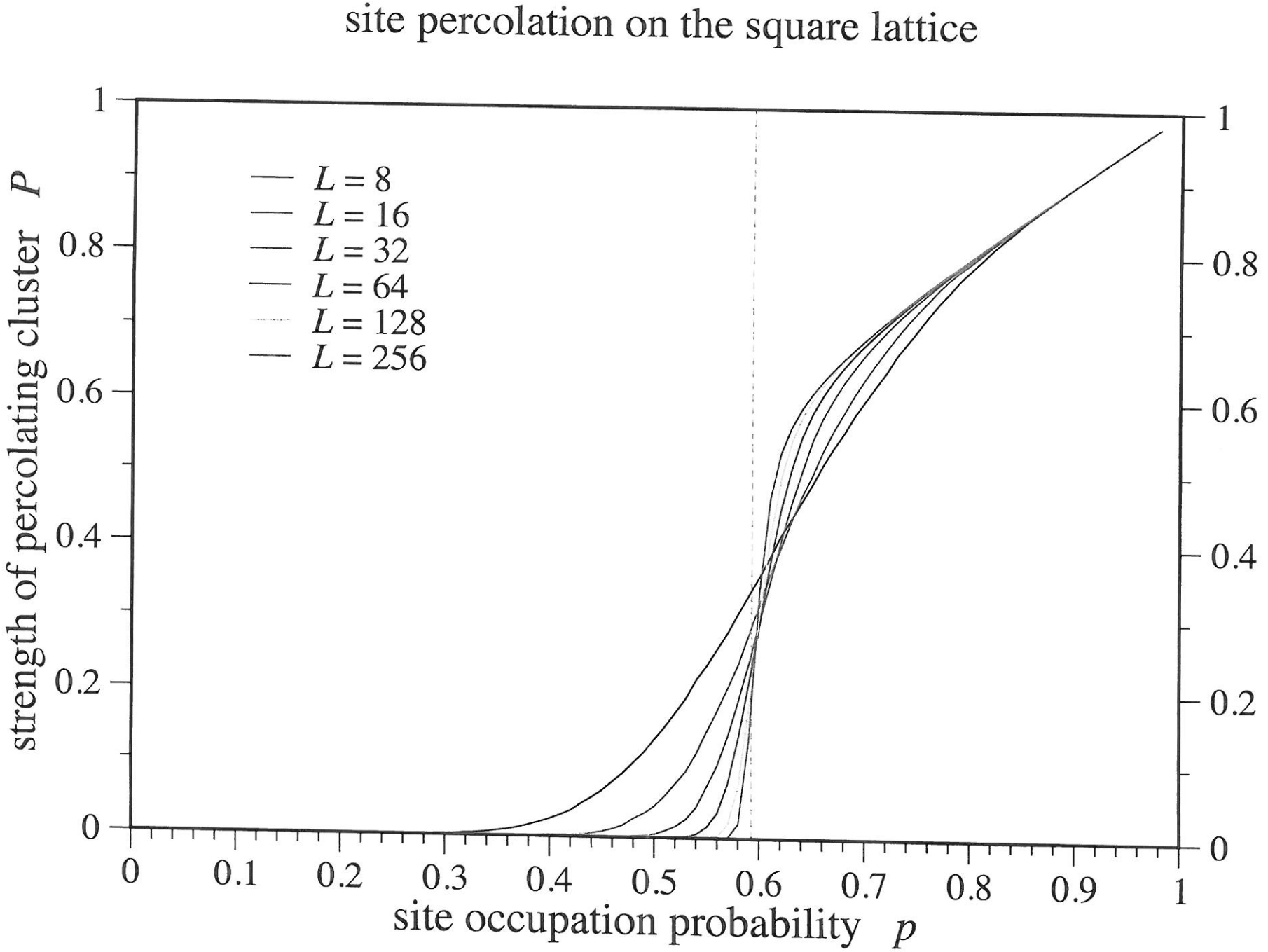
Finally, we obtain by combination of the previous scaling laws

$$\boxed{d\nu = \gamma + 2\beta = 2 - \alpha}$$

Those scaling laws involving dimensions (d) are usually called "hyper scaling" laws.

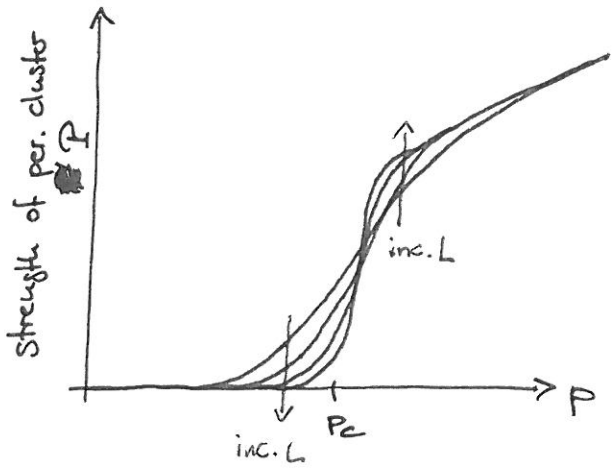
While the other scaling laws hold for all dimensions the hyper scaling law will break down for large d – e.g. it is clearly not valid for the Bethe lattice.

It holds up to the "upper critical dimension" $d_{upper} = 6$ in the case of percolation. For $d \geq d_{upper}$ the exponents are always the same as those of the infinite dimensional Bethe lattice.


site percolation on the square lattice
strength of percolating cluster P
site occupation probability p
$L = 8$
$L = 16$
$L = 32$
$L = 64$
$L = 128$
$L = 256$
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5
0.7
0.9

Finite-size effects

We now want to consider results obtained e.g. by Monte Carlo sampling on finite lattices.



- It can be seen that for $p \ll p_c$ P converges rapidly to zero, as L is increased.
- For $p \gg p_c$ on the other hand it converges rapidly to a finite value, as expected since P is the probability of a site belonging to the percolating cluster, which exists for $p > p_c$.
- We can also see that as L is increased a singularity starts to develop at p_c and it is plausible that P follows a power law $\boxed{P \propto (p-p_c)^{\beta}}$ in the infinite system.

- However, we also see that convergence is slow around p_c and we have no good way to directly estimate p_c nor the exponents.

The problem is that the correlation length ξ diverges as $\boxed{\xi \propto |p-p_c|^{-\nu}}$ and we would need lattices $L \gg \xi \to \infty$, which is impossible.

- We thus need a new idea.

The new idea is to extend the scaling ansatz so that it also contains the system size L as a parameter.

A finite system size L introduces a cutoff for all length scales. Then on finite systems ξ cannot grow larger than L.

This means that a finite system size L has the same effect as a finite distance from the critical point, related by

$$L \sim \xi \propto (p - p_c)^{-\nu} \qquad |p - p_c| \propto \xi^{-1/\nu}$$

Thus, for example, at the critical point

$$P(L) \simeq (p - p_c)^{\beta} \propto L^{-\beta/\nu} .$$

- More formally we can make the following scaling ansatz. Close to criticality the only important length scale is ξ. The effects of finite systems are thus determined only by the ratio L/ξ.
For a quantity X, which diverges as $X \propto (p - p_c)^{-\chi}$ for $L \gg \xi$ we make the ansatz

$$X(L, p) = (p - p_c)^{-\chi} \tilde{F}_1(L/\xi)$$

$$\left[= (p - p_c)^{-\chi} F_1((p - p_c) L^{1/\nu}) \right]$$

or equivalently

$$X(L, \xi) = \xi^{\chi/\nu} F_2(L/\xi) \propto \begin{cases} \xi^{\chi/\nu} & \text{for } L \gg \xi \\ L^{\chi/\nu} & \text{for } \xi \gg L \end{cases}$$

Applying this scaling ansatz to the size of the percolating cluster $L^d \cdot P$ we immediately arrive at

$$L^{D} = L^d \cdot P(L, \xi) = L^d \, \xi^{-\beta/\nu} \, f(L/\xi)$$

(D: fract. dim.)

$$\propto L^{d-\beta/\nu}$$

where we have chosen $L = \text{const} \times \xi$, allowing us to replace ξ by L, and giving a constant value for the scaling function $f(L/\xi)$.

- Thus we see that <u>finite-size scaling</u> allows us to determine the <u>ratio of exponents like β/ν</u> by calculating the L-dependence of P at $p = p_c$.

- One problem however still remains: How can we **determine p_c** on a finite lattice?

The answer is again finite-size scaling. Consider the <u>probability $\Pi(p)$ for the existence of a percolating cluster</u>. For the infinite system

$$\Pi(p) = \Theta(p - p_c)$$

For a finite system the step function is smeared out.

We can make the usual finite-size scaling ansatz

$$\Pi(p, L) = \Phi\left((p - p_c) L^{1/\nu}\right) \qquad (*)$$

(Φ: $X = 0$)

The derivative $d\Pi/dp$ gives the probability for first finding a percolating cluster at concentrations in the interval $[p, p+dp[$. The probability p at which a percolating cluster appears can easily be measured in a Monte Carlo simulation. Its average

$$p_{av} = \int p \cdot \frac{d\Pi}{dp} \, dp$$

is slightly different from the exact value p_c on any finite lattice, but converges like

$$p_{av} - p_c \propto L^{-1/\nu}$$

as can be seen by integrating the scaling ansatz (*).

Similarly, the variance

$$\Delta^2 = \int (\tilde{p} - p_{av})^2 \frac{d\Pi}{dp} \, dp$$

also decreases as

$$\Delta \propto L^{-1/\nu}$$

Thus, we can obtain ν and p_c from the finite-size scaling of the average p at which a percolating cluster appears on a finite lattice.

Binary search for average p → exercises.

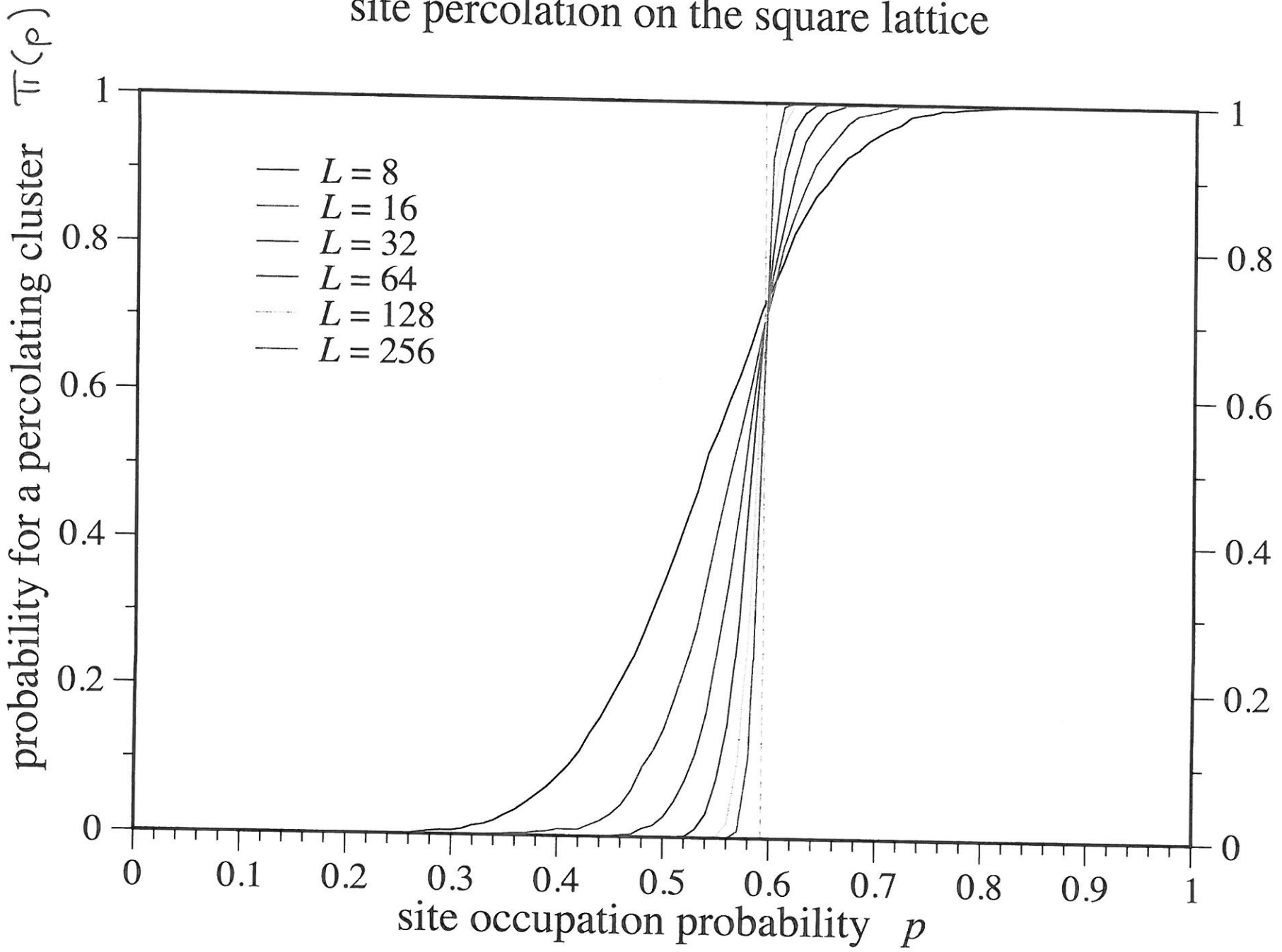

site percolation on the square lattice
probability for a percolating cluster $\Pi(p)$
site occupation probability p
$L = 8$
$L = 16$
$L = 32$
$L = 64$
$L = 128$
$L = 256$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

Table 2.1: Critical exponents for percolation in any dimension

functional form	exponent	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d \geq 6$
$M_0 \propto \lvert p-p_c \rvert^{2-\alpha}$	α	1	-2/3	-0.62	-0.72	-0.86	- 1
$n_s(p=p_c) \propto s^{-\tau}$	τ	2	187/91	2.18	2.31	2.41	5/2
$S \propto \lvert p-p_c \rvert^{-\gamma}$	γ	1	43/18	1.80	1.44	1.18	1
$\xi \propto \lvert p-p_c \rvert^{-\nu}$	ν	1	4/3	0.88	0.68	0.57	1/2
$g(r,p=p_c) \propto r^{-(d-2+\eta)}$	η	1	5/24	-0.07	-0.12	-0.05	0
$P \propto (p-p_c)^{\beta}$	β	–	5/36	0.41	0.64	0.84	1
fractal dimension	D	1	91/48	2.53	3.06	3.54	4

Finally we wish to repeat that these exponents are universal in the sense that they depend only on dimensionality, but not on the specific percolation model used. Thus all the examples mentioned in the introduction to this chapter: forest fires, oil reservoirs, gelation, spreading of diseases, etc., share the same exponents as long as the dimensionality is the same.

Thus, although we have considered only toy models we get widely applicable results. This is a common thread that we will encounter again. While performing numerical simulations for specific models is nice, being able to extract universally valid results from specific model simulations is the high art of computer simulations.