

KITAEV HONEYCOMB MODEL [1]

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THE MODEL

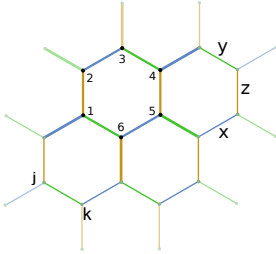


Figure 1: The Lattice

The model consists of a honeycomb lattice with a spin sitting on each of its vertices. The spins interact with their nearest neighbors via three different types of links. The x-, y- and z-interaction. The interaction of the spins can be described using the pauli spin operators σ_i^α , where $\alpha = x, y, z$ and i indexing the site.

The Hamiltonian has the following form:

$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$

In the lattice we can define a plaquette(hexagon) and the operator $W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$ which commutes with the Hamiltonian and itself. Thus, the Hamiltonian can be solved individually for the eigenspaces of W_p . The original Hilberspace is of the dimension 2^n where n is the number of lattice-points. The use of the eigenspaces only reduces our problem to the dimension of $2^{n/2}$. It turns out, that if we represent our spin operators with majorana operators we can simplify our model even more and get the Hamiltonian to a quadratic form.

REPRESENTING SPINS BY MAJORANA FERMIONS

A system with n fermionic modes is usually described by the annihilation and creation operators a_k and a_k^\dagger . Instead, one can use their linear combinations $c_{2k-1} = a_k + a_k^\dagger$ and $c_{2k} = \frac{a_k - a_k^\dagger}{i}$ which are called majorana operators. We now represent our spins by two fermionic modes, i.e. by four majorana operators.

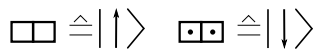


Figure 2: Majorana representation of spin up and spin down

We choose the state with no fermions to be spin up and the state with two fermions to be spin down. The states with one fermion dont have any physical relevance to us.

In the following we will use the following notation: $b^x = c_1$, $b^y = c_2$, $b^z = c_3$, $c = c_4$. It turns out, that we can use an operator $D = b^x b^y b^z c$ to determine whether any state $|\psi\rangle$ is physical or not. $|\psi\rangle$ is physical $\leftrightarrow D|\psi\rangle = |\psi\rangle$. The pauli operators $\sigma^x \sigma^y \sigma^z$ can be represented by $\tilde{\sigma}^x = ib^y c$, $\tilde{\sigma}^y = ib^x c$, $\tilde{\sigma}^z = ib^z c$ which act on the extended space.

APPLICATION TO THE MODEL

If we replace the pauli operators with those that act on the extended space we get $\tilde{\sigma}_j^\alpha \tilde{\sigma}_k^\alpha = (ib_j^\alpha c_j)(ib_k^\alpha c_k) = -i u_{jk} c_j c_k$ with $u_{jk} = ib_j^\alpha b_k^\alpha$ which we associate with the link (j,k) . We now insert this into the Hamiltonian and get

$$H = \frac{i}{4} \sum_{j,k} \hat{A}_{jk} c_j c_k$$

with $\hat{A}_{jk} = 2J_{\alpha_{jk}} u_{jk}$ if (j,k) are connected and $\hat{A}_{jk} = 0$ else. Remarkably, the operators \hat{A}_{jk} commute with the Hamiltonian and with each other and have the eigenvalues ± 1 . Remember the operators W_p did the same. Using a theorem called "Lieb's Theorem", we know that the groundstate of the system lies in the subspace where all operators W_p have the eigenvalue +1 (vortex free configuration). This leads to the fact that we can replace the operators \hat{A}_{jk} by the eigenvalue +1 - a procedure Kitaev calls "removing hats". From this we finally obtain our Hamiltonian in the quadratic form $H = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k$ where A is no more an operator but a number $A_{jk} = 2J_{\alpha_{jk}}$. It turns out, that the configuration $u_{jk} = +1$ is translational invariant and we can solve our problem using Fourier-Transformation. We will represent the site index j as (s, λ) where s refers to a unit cell and λ to a position inside the cell. The calculation leads to the Fourier transformed Hamiltonian:

$$H = \frac{1}{2} \sum_{q, \lambda, \mu} i \tilde{A}_{\lambda, \mu}(\vec{q}) a_{-\vec{q}, \lambda} a_{\vec{q}, \mu}$$

with the matrix

$$i \tilde{A}(\vec{q}) = \begin{pmatrix} 0 & i f(\vec{q}) \\ -i f(\vec{q})^* & 0 \end{pmatrix}$$

and the function $f(\vec{q}) = 2(J_x e^{i(\vec{q}, \vec{n}_1)} + J_y e^{i(\vec{q}, \vec{n}_2)} + J_z)$. Out of this we get the energy dispersion

$$\epsilon(\vec{q}) = \pm |f(\vec{q})|$$

An important property of this function is, whether it has zeros for some \vec{q} or not. We say it is gapless if it has zeros

and say gapped if not. The equation $f(\vec{q}) = 0$ only has solutions if and only if $|J_x|, |J_y|, |J_z|$ satisfy the triangle inequalities. That means if the parameters are in the range of phase B in figure 3.

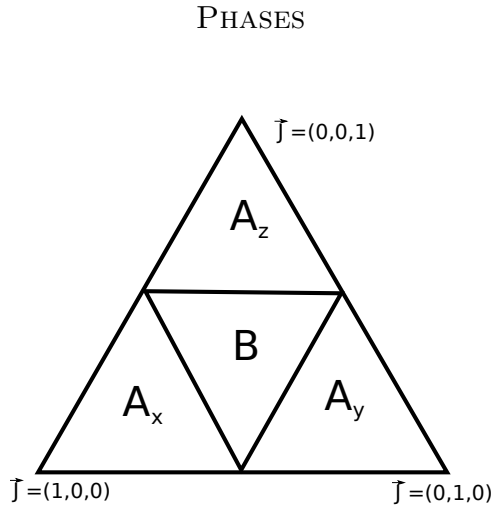


Figure 3: *Parameter space of the coupling strengths*

Let us now consider the plane $J_x + J_y + J_z = 1$ in the parameter space. There are three equivalent gapped phases called phase A here and one gapless phase B.

In phase A there is only one dominant interaction, while the other two types are weak. This leads to a simpler model, where pairs of spins live on a lattice of disjoint dimers. By adjusting the unit cell to a square lattice, this model can be reduced to the toric code [2].

In contrast to the described case there is also a highly frustrated case, where all the coupling parameters are roughly the same, which takes place in the gapless phase B. In this phase the system corresponds to a quantum spin liquid with \mathbb{Z}_2 topological order, which is disordered even at lowest temperature due to quantum fluctuation.

REFERENCES

- [1] A. Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2 – 111, 2006.
- [2] A. Kitaev and A. Laumann. Topological phases and quantum computation.